# 3.4 Transformations of Cubic and Quartic Functions

**GOAL**

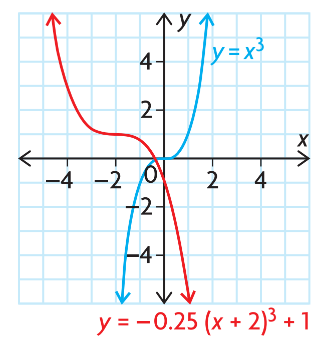
Describe and perform transformations on cubic and quartic functions.

**YOU WILL NEED**

- graphing calculator or graphing software

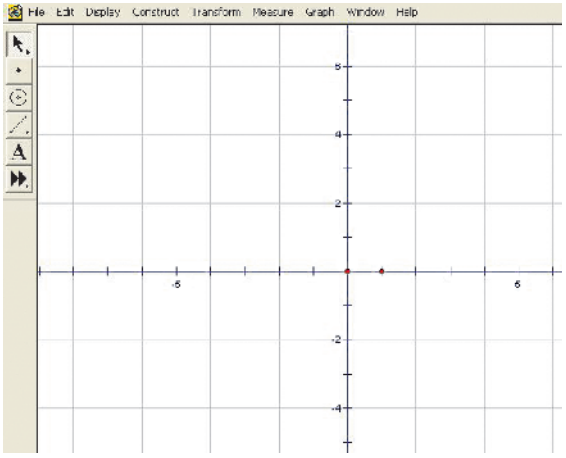
## *INVESTIGATE the Math*

The graphs of the parent cubic function $y = x^3$ and a second function, which is a transformation of the parent function, are shown.

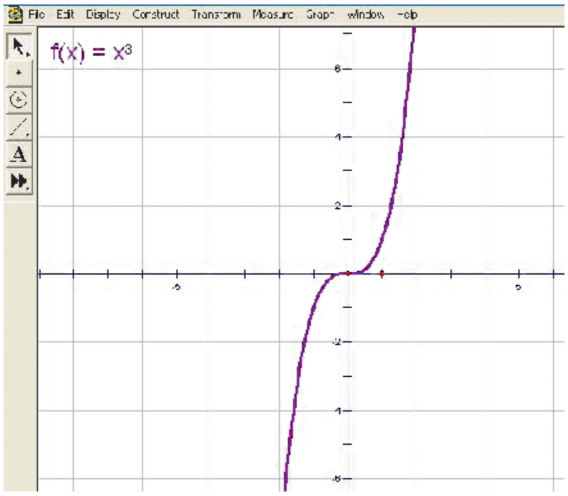



**?** How do the graphs of $y = a(k(x - d))^3 + c$ and $y = a(k(x - d))^4 + c$ relate to the graphs of $y = x^3$ and $y = x^4$?

**A.** Use dynamic geometry software to create a Cartesian grid with an $x$-axis and $y$-axis.



**B.** Plot $f(x) = x^3$.

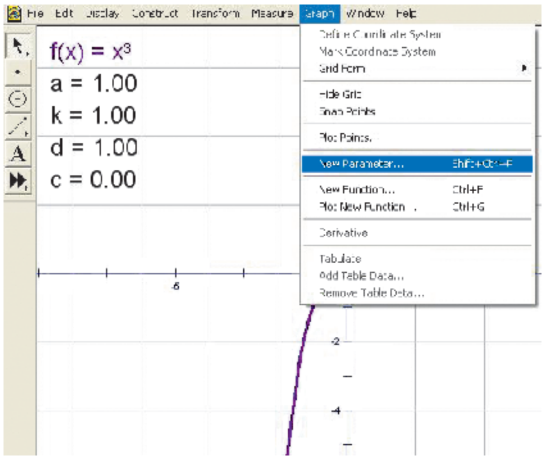



**Tech** ***Support***

For information about how to use *The Geometer's Sketchpad* to plot functions, see Technical Appendix, T-19.

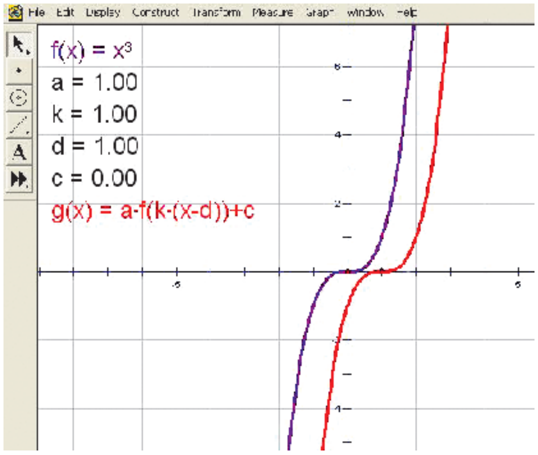
**C.** Define four new parameters: $a = 1$, $k = 1$, $d = 1$, and $c = 0$.

**D.** Create and plot $g(x) = a(k(x - d))^3 + c$. Describe how the new graph, $g(x)$, is related to the graph of the parent function, $f(x)$.




**E.** Make a conjecture about how changing the parameter $a$ in the function $g(x)$ will affect the graph of the parent function, $f(x)$.

**F.** Change the value of $a$ at least four times using integers and rational numbers. Record the effect of each change on the graph. Make sure that you use both positive and negative values.




**G.** Repeat parts E and F for each of the other parameters $(k, d, \text{and } c)$.

**H.** Repeat parts A to G for the quartic function $f(x) = x^4$.

## Reflecting

**I.** Describe the transformations that must be applied to the graph of the function $f(x) = x^3$ to create the graph of $y = a(k(x - d))^3 + c$.

**J.** Describe the transformations that must be applied to the graph of the function $f(x) = x^4$ to create the graph of $y = a(k(x - d))^4 + c$.

**K.** Do you think your descriptions in parts I and J can be applied to transformations of the function $f(x) = x^n$ for all possible values of $n$? Explain.

## *APPLY* the Math

### EXAMPLE 1 Using reasoning to determine transformations

Describe the transformations that must be applied to $y = x^3$ to graph $y = -8\left(\frac{1}{2}x + 1\right)^3 - 3$, and then graph this function.

#### Solution A: Using the equation as given

$$y = -8\left(\frac{1}{2}x + 1\right)^3 - 3$$

$$y = -8\left(\frac{1}{2}(x + 2)\right)^3 - 3$$

Factor the coefficient of $x$ so that the function is in the form $y = a(k(x - d))^3 + c$.

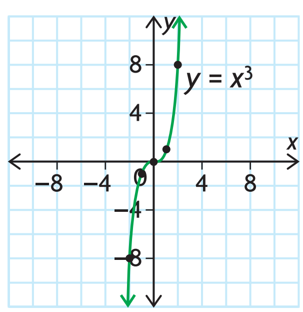
$y = x^3$ is

- vertically stretched by a factor of 8 and reflected in the $x$-axis
- horizontally stretched by a factor of 2
- translated 2 units left
- translated 3 units down

$a = -8$
$k = \frac{1}{2}$
$d = -2$
$c = -3$

$y = x^3$

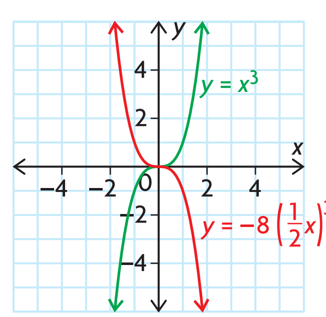
| $x$ | $y$ |
|---|---|
| $-2$ | $-8$ |
| $-1$ | $-1$ |
| 0 | 0 |
| 1 | 1 |
| 2 | 8 |




Begin with the parent function to be transformed and its key points.

$(x, y) \rightarrow (2x, -8y)$

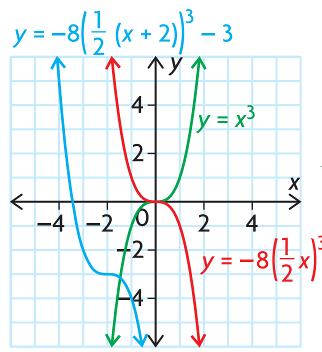
| $y = x^3$ | $y = -8\left(\frac{1}{2}x\right)^3$ |
|---|---|
| $(-2, -8)$ | $(2(-2), -8(-8)) = (-4, 64)$ |
| $(-1, -1)$ | $(2(-1), -8(-1)) = (-2, 8)$ |
| $(0, 0)$ | $(2(0), -8(0)) = (0, 0)$ |
| $(1, 1)$ | $(2(1), -8(1)) = (2, -8)$ |
| $(2, 8)$ | $(2(2), -8(8)) = (4, -64)$ |




Perform the stretches, reflections, and compressions first. Multiply the $x$-coordinates of the key points by 2. Multiply the $y$-coordinates of the key points by $-8$.

$(2x, -8y) \rightarrow (2x - 2, -8y - 3)$

| $y = -8\left(\frac{1}{2}x\right)^3$ | $y = -8\left(\frac{1}{2}(x + 2)\right)^3 - 3$ |
|---|---|
| $(-4, 64)$ | $(-4 - 2, 64 - 3) = (-6, 61)$ |
| $(-2, 8)$ | $(-2 - 2, 8 - 3) = (-4, 5)$ |
| $(0, 0)$ | $(0 - 2, 0 - 3) = (-2, -3)$ |
| $(2, -8)$ | $(2 - 2, -8 - 3) = (0, -11)$ |
| $(4, -64)$ | $(4 - 2, -64 - 3) = (2, -67)$ |




Perform the translations last. Subtract 2 from the $x$-coordinate and 3 from the $y$-coordinate of each point on the red graph to obtain the corresponding point on the blue graph.

## Solution B: Simplifying the equation first

$$y = -8\left(\frac{1}{2}x + 1\right)^3 - 3$$

$$y = -8\left(\frac{1}{2}(x + 2)\right)^3 - 3$$

$$y = -8\left(\frac{1}{2}\right)^3(x + 2)^3 - 3$$

$$y = -(x + 2)^3 - 3$$
$$y = -(x + 2)^3 - 3$$

Factor out the coefficient of $x$, and apply the exponent to both parts of the product.

Simplify.

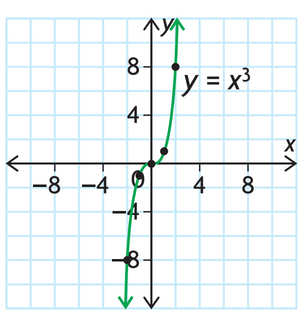
$y = x^3$ is

- vertically reflected in the $x$-axis
- translated 2 units to the left
- translated 3 units down

$a = -1$
$d = -2$
$c = -3$

$y = x^3$

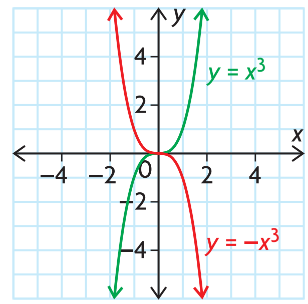
| $x$ | $y$ |
|---|---|
| $-2$ | $-8$ |
| $-1$ | $-1$ |
| $0$ | $0$ |
| $1$ | $1$ |
| $2$ | $8$ |




Begin with the graph of the parent function to be transformed.

$(x, y) \rightarrow (x, -y)$

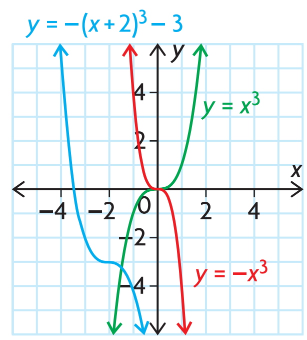
| $y = x^3$ | $y = -x^3$ |
|---|---|
| $(-2, -8)$ | $(-2, -(-8)) = (-2, 8)$ |
| $(-1, -1)$ | $(-1, -(-1)) = (-1, 1)$ |
| $(0, 0)$ | $(0, -(0)) = (0, 0)$ |
| $(1, 1)$ | $(1, -(1)) = (1, -1)$ |
| $(2, 8)$ | $(2, -(8)) = (2, -8)$ |




Apply the reflection in the *x*-axis first. Multiply the *y*-coordinate of each key point on the green graph to obtain the corresponding point on the red graph.

$(x, -y) \rightarrow (x - 2, -y + 3)$

| $y = -x^3$ | $y = -(x + 2)^3 - 3$ |
|---|---|
| $(-2, 8)$ | $(-2 - 2, 8 - 3) = (-4, 5)$ |
| $(-1, 1)$ | $(-1 - 2, 1 - 3) = (-3, -2)$ |
| $(0, 0)$ | $(0 - 2, 0 - 3) = (-2, -3)$ |
| $(1, -1)$ | $(1 - 2, -1 - 3) = (-1, -4)$ |
| $(2, -8)$ | $(2 - 2, -8 - 3) = (0, -11)$ |




Apply the translations last. Subtract 2 from the *x*-coordinate and 3 from the *y*-coordinate of each point on the red graph to obtain the corresponding point on the blue graph.

Note that the final graph, shown in blue, is the same from the two different solutions shown above. Two different sets of transformations have resulted in the same final graph.

## EXAMPLE 2 Selecting a strategy to determine the roots of a quartic function

Determine the *x*-intercept(s) of the function $y = 3(x + 6)^4 - 48$.

### Solution A: Using algebra

$y = 3(x + 6)^4 - 48$

$0 = 3(x + 6)^4 - 48$

Let $y = 0$, and solve for $x$.

$48 = 3(x + 6)^4$

$\frac{48}{3} = \frac{3(x + 6)^4}{3}$

Use inverse operations to isolate $(x + 6)^4$. Add 48 to both sides, and then divide both sides by 3.

$16 = (x + 6)^4$

$\pm\sqrt[4]{16} = \sqrt[4]{(x + 6)^4}$

Take the fourth root of both sides.

$\pm 2 = x + 6$

Any even root of a number has both a positive value and a negative value.

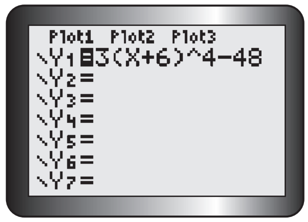
$2 = x + 6$ and $-2 = x + 6$

$2 - 6 = x \qquad -2 - 6 = x$

$-4 = x \qquad -8 = x$

The *x*-intercepts are $-4$ and $-8$.

## Solution B: Using transformations and a graphing calculator




Enter the equation into a graphing calculator.

To graph $y = 3(x + 6)^4 - 48$, $y = x^4$ must be

- vertically stretched by a factor of 3 since $a = 3$
- translated 6 units to the left and 48 units down, since $d = -6$ and $c = -48$

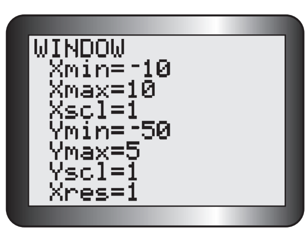
Use transformations to help determine suitable window settings.

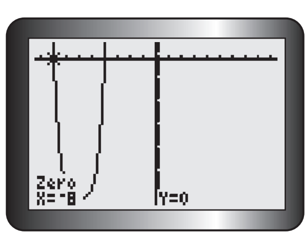
$$
\begin{aligned}
(x, y) &\rightarrow (x - 6, 3y - 48)\\
(0, 0) &\rightarrow (0 - 6, 3(0) - 48)\\
&= (-6, -48)
\end{aligned}
$$

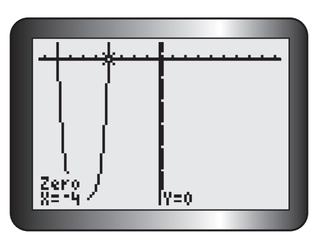
Determine the new location of the turning point, $(0, 0)$, of the parent function. Subtract 6 from the $x$-coordinate. Multiply the $y$-coordinate by 3, and subtract 48.

Since $a > 0$, the graph opens up.

Adjust the window settings so that Xmin is to the left of $x = -6$ and Ymin is below $y = -48$.




Graph the function. Use the zero operation to determine the locations of the zeros.

The $x$-intercepts are $-8$ and $-4$.

**Tech Support**

For help using the zero operation on a graphing calculator to determine the zeros of a function, see Technical Appendix, T-8.

## In Summary

### Key Ideas

- The polynomial function $y = a(k(x - d))^n + c$ can be graphed by applying transformations to the graph of the parent function $y = x^n$, where $n \in \mathbf{N}$. Each point $(x, y)$ on the graph of the parent function changes to $\left(\frac{x}{k} + d, ay + c\right)$.
- When using transformations to graph a function in the fewest steps, you can apply $a$ and $k$ together, and then $c$ and $d$ together.

### Need to Know

- In $y = a(k(x - d))^n + c$,
  - the value of $a$ represents a vertical stretch/compression and possibly a vertical reflection
  - the value of $k$ represents a horizontal stretch/compression and possibly a horizontal reflection
  - the value of $d$ represents a horizontal translation
  - the value of $c$ represents a vertical translation

## *CHECK Your Understanding*

**1.** Match each function with the most likely graph. Explain your reasoning.

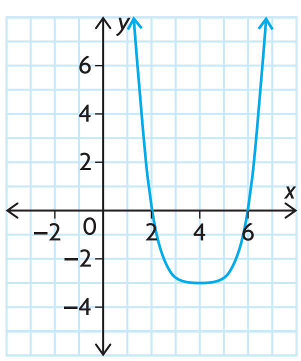
**a)** $y = 2(x - 3)^3 + 1$

**b)** $y = -\frac{1}{3}(x + 1)^3 - 1$

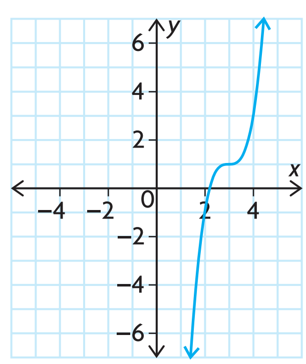
**c)** $y = 0.2(x - 4)^4 - 3$

**d)** $y = -1.5(x + 3)^4 + 4$

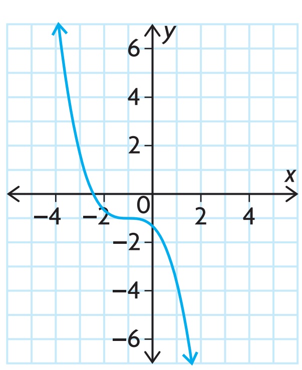
**A**



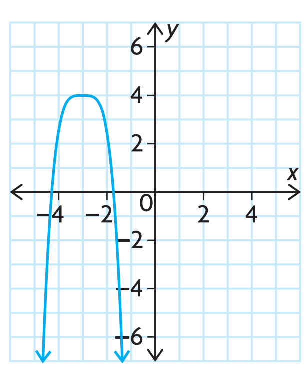

**B**




**C**




**D**



2. State the parent function that must be transformed to create the graph of each of the following functions. Then describe the transformations that must be applied to the parent function.

a) $y = \frac{5}{4}x^4 + 3$

b) $y = 3x - 4$

c) $y = (3x + 4)^3 - 7$

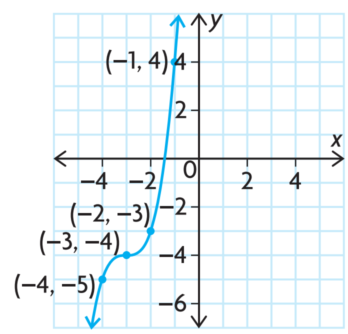
d) $y = -(x + 8)^4$

e) $y = -4.8(x - 3)(x - 3)$

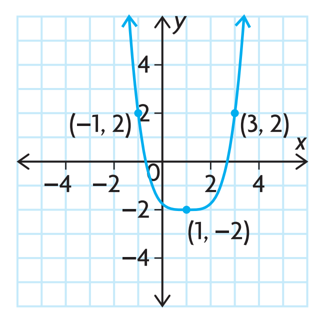
f) $y = 2\left(\frac{1}{5}x + 7\right)^3 - 4$

3. Describe the transformations that were applied to the parent function to create each of the following graphs. Then write the equation of the transformed function.

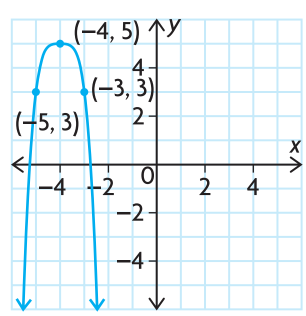
a) parent function: $y = x^3$



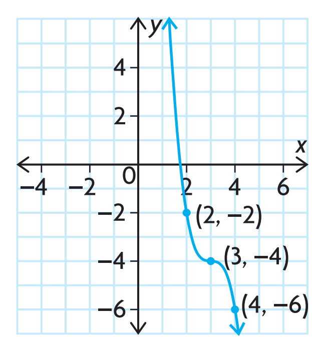

c) parent function: $y = x^4$




b) parent function: $y = x^4$




d) parent function: $y = x^3$




## PRACTISING

4. **K** Describe the transformations that were applied to $y = x^3$ to create each of the following functions.

a) $y = 12(x - 9)^3 - 7$

b) $y = \left(\frac{7}{8}(x + 1)\right)^3 + 3$

c) $y = -2(x - 6)^3 - 8$

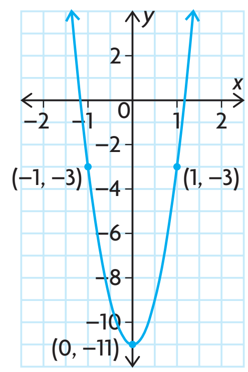
d) $y = (x + 9)(x + 9)(x + 9)$

e) $y = -2(-3(x - 4))^3 - 5$

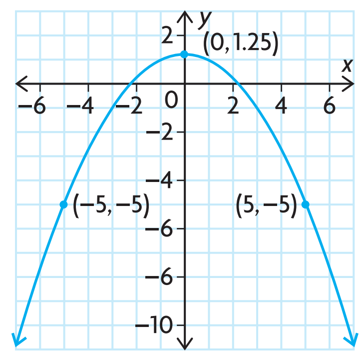
f) $y = \left(\frac{3}{4}(x - 10)\right)^3$

**5.** For each graph, determine the equation of the function in the form $y = a(x - h)^2 + k$. Then describe the transformations that were applied to $y = x^2$ to obtain each graph.

**a)**




**b)**




**6.** The function $y = x^3$ has undergone the following sets of transformations. If $y = x^3$ passes through the points $(-1, -1)$, $(0, 0)$, and $(2, 8)$, list the coordinates of these transformed points on each new curve.

**a)** vertically compressed by a factor of $\frac{1}{2}$, horizontally compressed by a factor of $\frac{1}{5}$, and horizontally translated 6 units to the left

**b)** reflected in the $y$-axis, horizontally stretched by a factor of 2, and vertically translated 3 units up

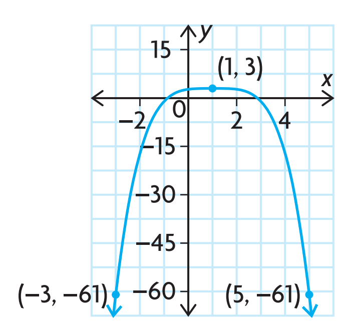
**c)** reflected in the $x$-axis, vertically stretched by a factor of 3, horizontally translated 4 units to the right, and vertically translated $\frac{1}{2}$ of a unit down

**d)** vertically compressed by a factor of $\frac{1}{10}$, horizontally stretched by a factor of 7, and vertically translated 2 units down

**e)** reflected in the $y$-axis, reflected in the $x$-axis, and vertically translated $\frac{9}{10}$ of a unit up

**f)** horizontally stretched by a factor of 7, horizontally translated 4 units to the left, and vertically translated 2 units down

**7.** T The graph shown is a result of transformations applied to $y = x^4$. Determine the equation of this transformed function.




**8.** A Dikembe has reflected the function $g(x) = x^3$ in the $x$-axis, vertically compressed it by a factor of $\frac{2}{3}$, horizontally translated it 13 units to the right, and vertically translated it 13 units down. Three points on the resulting curve are $\left(11, -\frac{23}{3}\right)$, $(13, -13)$, and $\left(15, -\frac{55}{3}\right)$. Determine the original coordinates of these three points on $g(x)$.

**9.** Determine the $x$-intercepts of each of the following polynomial functions. Round to two decimal places, if necessary.

a) $y = 2(x + 3)^4 - 2$
b) $y = (x - 2)^3 - 8$
c) $y = -3(x + 1)^4 + 48$
d) $y = -5(x + 6)^4 - 10$
e) $y = 4(x - 8)^4 - 12$
f) $y = -(2x + 5)^3 - 20$

**10.** Consider the function $y = 2(x - 4)^n + 1$, $n \in \mathbf{N}$.

a) How many zeros will the function have if $n = 3$? Explain how you know.
b) How many zeros will the function have if $n = 4$? Explain how you know.
c) Make a general statement about the number of zeros that the function will have, for any value of $n$. Explain your reasoning.

**11.** a) For what values of $n$ will the reflection of the function $y = x^n$ in the $x$-axis be the same as its reflection in the $y$-axis. Explain your reasoning.

b) For what values of $n$ will the reflections be different? Explain your reasoning.

**12.** Consider the function $y = x^3$.

C a) Use algebraic and graphical examples to describe all the transformations that could be applied to this function.
b) Explain why just creating a single table of values is not always the best way to sketch the graph of a function.

## Extending

**13.** Can you create the graph of the function $y = 2(x - 1)(x + 4)(x - 5)$ by transforming the function $y = (x - 4)(x + 1)(x - 8)$? Explain.

**14.** Transform the graph of the function $y = (x - 1)^2(x + 1)^2$ to determine the roots of the function $y = 2(x - 1)^2(x + 1)^2 + 1$.

**15.** The function $f(x) = x^2$ was transformed by vertically stretching it, horizontally compressing it, horizontally translating it, and vertically translating it. The resulting function was then transformed again by reflecting it in the $x$-axis, vertically compressing it by a factor of $\frac{4}{5}$, horizontally compressing it by a factor of $\frac{1}{2}$, and vertically translating it 6 units down. The equation of the final function is $f(x) = -4(4(x + 3))^2 - 5$. What was the equation of the function after it was transformed the first time?