# 3.5 Dividing Polynomials

**GOAL**

Use a variety of strategies to determine the quotient when one polynomial is divided by another polynomial.

## *LEARN ABOUT the Math*

Recall that long division can be used to determine the quotient of two numbers. For example, 107 ÷ 4 can be evaluated as follows:

$$\begin{array}{rrl} & 26 & \leftarrow \text{quotient} \\ \text{divisor} \rightarrow 4 & \overline{)107} & \leftarrow \text{dividend} \\ & \underline{8\downarrow} & \\ & 27 & \\ & \underline{24} & \\ & 3 & \leftarrow \text{remainder} \end{array}$$

Every division statement that involves numbers can be rewritten using multiplication and addition. The multiplication is the quotient, and the addition is the remainder. For example, since $107 = (4)(26) + 3$, then $\frac{107}{4} = 26 + \frac{3}{4}$. The quotient is 26, and the remainder is 3.

**?** How can you use a similar strategy to determine the quotient of $(3x^3 - 5x^2 - 7x - 1) \div (x - 3)$?

**EXAMPLE 1** Selecting a strategy to divide a polynomial by a binomial

Determine the quotient of $(3x^3 - 5x^2 - 7x - 1) \div (x - 3)$.

### Solution A: Using polynomial division

$$x - 3\overline{)3x^3 - 5x^2 - 7x - 1}$$

Focus on the *first terms* of the dividend and the divisor, and then determine the quotient when these terms are divided. Here, the first term of the dividend is $3x^3$ and the first term of the divisor is $x$.

$$\begin{array}{r} 3x^2 \phantom{{}- 5x^2 - 7x - 1} \\ x - 3\overline{)3x^3 - 5x^2 - 7x - 1} \end{array}$$

Since $3x^3 \div x = 3x^2$, this becomes the first term of the quotient. Place $3x^2$ above the term of the dividend with the same degree.

$$\begin{array}{r|l} & 3x^2 \\ \hline x-3 & 3x^3 - 5x^2 - 7x - 1 \\ & \underline{3x^3 - 9x^2} \\ & \quad\;\; 4x^2 \end{array}$$

Multiply $3x^2$ by the divisor, and write the answer below the dividend. Make sure that you line up "like terms." $3x^2(x - 3) = 3x^3 - 9x^2$. Subtract this product from the dividend.

$$\begin{array}{r|l} & 3x^2 + 4x \\ \hline x-3 & 3x^3 - 5x^2 - 7x - 1 \\ & \underline{-3x^3 + 9x^2} \quad \downarrow \\ & \qquad\; 4x^2 - 7x \\ & \qquad\; \underline{4x^2 - 12x} \\ & \qquad\qquad\quad 5x \end{array}$$

Now focus on $x$ in the divisor $x - 3$ and $4x^2$ in the expression $4x^2 - 7x$. Determine the quotient when these terms are divided. Since $4x^2 \div x = 4x$, place $4x$ above the $x$ in the dividend. Multiply $4x$ by the divisor. Write the answer below the last line (making sure that you line up like terms), and then subtract.

$$\begin{array}{rr|l} & & 3x^2 + 4x + 5 \\ \hline & x-3 & 3x^3 - 5x^2 - 7x - 1 \\ 3x^2(x-3) \rightarrow & & \underline{3x^3 - 9x^2} \quad \downarrow \quad \downarrow \\ & & \qquad\; 4x^2 - 7x \\ 4x(x-3) \rightarrow & & \qquad\; \underline{4x^2 - 12x} \\ & & \qquad\qquad\quad 5x - 1 \\ 5(x-3) \rightarrow & & \qquad\qquad\quad \underline{5x - 15} \\ & & \qquad\qquad\qquad\; 14 \end{array}$$

Repeat this process until the degree of the remainder is less than the degree of the divisor.

Since the divisor has degree 1, the remainder should be a constant.

$$\begin{aligned} &3x^3 - 5x^2 - 7x - 1 \\ &\quad = (x - 3)(3x^2 + 4x + 5) + 14 \end{aligned}$$

Write the multiplication statement that shows how the divisor, dividend, quotient, and remainder are all related.

$$\begin{aligned} &(x - 3)(3x^2 + 4x + 5) + 14 \\ &\quad = 3x^3 + 4x^2 + 5x - 9x^2 \\ &\qquad - 12x - 15 + 14 \\ &\quad = 3x^3 - 5x^2 - 7x - 1 \end{aligned}$$

To check, expand and simplify the right side of the division statement.

The result is the dividend, which confirms the division was done correctly.

## Solution B: Using synthetic division

$(3x^3 - 5x^2 - 7x - 1) \div (x - 3) \rightarrow k = 3$

$$\begin{array}{c|cccc} 3 & 3 & -5 & -7 & -1 \\ & & & & \\ \hline & & & & \end{array}$$

Synthetic division is an efficient way to divide a polynomial by a binomial of the form $(x - k)$.

Create a chart that contains the coefficients of the dividend, as shown. The dividend and binomial must be written with its terms in descending order, by degree.

$$\begin{array}{c|cccc} 3 & 3 & -5 & -7 & -1 \\ & \downarrow & & & \\ & & 9 & & \\ \hline & 3 & & & \end{array}$$

$\times 3$

Bring the first term down. This is now the coefficient of the first term of the quotient.

Multiply it by $k$, and write the answer below the second term of the dividend.

$$\begin{array}{c|cccc} 3 & 3 & -5 & -7 & -1 \\ & & \text{add}\downarrow & & \\ & & 9 & & \\ \hline & 3 & 4 & & \end{array}$$

Now add the terms together.

$$\begin{array}{c|cccc} 3 & 3 & -5 & -7 & -1 \\ & & & \text{add}\downarrow & \\ & & 9 & 12 & \\ \hline & 3 & 4 & 5 & \end{array}$$

$\times 3$

Repeat this process for the answer you just obtained.

$$\begin{array}{c|cccc} 3 & 3 & -5 & -7 & -1 \\ & & & & \text{add}\downarrow \\ & & 9 & 12 & 15 \\ \hline & 3 & 4 & 5 & 14 \end{array}$$

$\times 3$

Repeat this process one last time.

$$\begin{array}{c|cccc} 3 & 3 & -5 & -7 & -1 \\ & & & & \\ & & 9 & 12 & 15 \\ \hline & 3 & 4 & 5 & 14 \end{array}$$

coefficients of quotient — remainder

$3x^2 + 4x + 5$

The last number below the chart is the remainder. The first numbers are the coefficients of the quotient, starting with the degree that is one less than the original dividend.

$$\begin{aligned} &3x^3 - 5x^2 - 7x - 1 \\ &\quad = (x - 3)(3x^2 + 4x + 5) + 14 \end{aligned}$$

Write the corresponding multiplication statement.

## Reflecting

**A.** When dividing an $n$th degree polynomial by a $k$th degree polynomial, what degree is the quotient? What degree is the remainder?

**B.** If you divide a number by another number and the remainder is zero, what can you conclude? Do you think you can make the same conclusion for polynomials? Explain.

**C.** If you had a divisor of $x + 5$, what value of $k$ would you use in synthetic division?

# *APPLY* the Math

**EXAMPLE 2** Selecting a strategy to determine the remainder in polynomial division

Determine the remainder of $\dfrac{5x - 2x^3 + 3 + x^4}{1 + 2x + x^2}$.

### Solution

$$x^2 + 2x + 1\overline{)x^4 - 2x^3 + 0x^2 + 5x + 3}$$

Write the terms of the dividend and the quotient in descending order, by degree.

Since there is no $x^2$-term in the dividend, use 0 as the coefficient of $x^2$ to make the like terms line up properly.

$$\begin{array}{r} x^2 \phantom{{}- 2x^3 + 0x^2 + 5x + 3} \\ x^2 + 2x + 1\overline{)\,x^4 - 2x^3 + 0x^2 + 5x + 3} \\ \underline{x^4 + 2x^3 + 1x^2} \phantom{{}+ 5x + 3} \\ -4x^3 - 1x^2 + 5x \phantom{{}+3} \end{array}$$

Follow the same steps that you use for long division with numbers.

$$\begin{array}{rr} & x^2 - 4x + 7 \\ x^2 + 2x + 1\overline{)} & x^4 - 2x^3 + 0x^2 + 5x + 3 \\ x^2(x^2 + 2x + 1) \rightarrow & \underline{x^4 + 2x^3 + 1x^2} \phantom{{}+ 5x + 3} \\ & -4x^3 - 1x^2 + 5x \phantom{{}+3} \\ -4x(x^2 + 2x + 1) \rightarrow & \underline{-4x^3 - 8x^2 - 4x} \phantom{{}+3} \\ & 7x^2 + 9x + 3 \\ 7(x^2 + 2x + 1) \rightarrow & \underline{7x^2 + 14x + 7} \\ & -5x - 4 \end{array}$$

Repeat this process until the degree of the remainder is less than the degree of the divisor.

Since the divisor was degree 2, the remainder should be degree 1.

The remainder is $-5x - 4$.

$$\begin{aligned} &x^4 - 2x^3 + 5x + 3 \\ &\quad = (x^2 + 2x + 1)(x^2 - 4x + 7) - 5x - 4 \end{aligned}$$

Write the corresponding division statement.

## EXAMPLE 3 Selecting a strategy to determine whether one polynomial is a factor of another polynomial

Determine whether $x + 2$ is a factor of $13x - 2x^3 + x^4 - 6$.

### Solution

Use synthetic division to divide $13x - 2x^3 + x^4 - 6$ by $x + 2$.

Rearrange the terms of the dividend in descending order. An $x^2$-term, with a coefficient of 0, needs to be added to the dividend.

$(x^4 - 2x^3 + 0x^2 + 13x - 6) \div (x + 2)$

$$\begin{array}{r|rrrrr} -2 & 1 & -2 & 0 & 13 & -6 \\ & \downarrow & -2 & 8 & -16 & 6 \\ \hline & 1 & -4 & 8 & -3 & 0 \end{array}$$

In this example, $k = -2$. Multiply and add to complete the chart.

$x + 2$ is a factor, and so is $x^3 - 4x^2 + 8x - 3$.

$$\begin{aligned} &(x + 2)(x^3 - 4x^2 + 8x - 3) \\ &\quad = x^4 - 2x^3 + 0x^2 + 13x - 6 \end{aligned}$$

The quotient is $x^3 - 4x^2 + 8x - 3$, and the remainder is 0. Since the remainder is 0, $x + 2$ must be a factor of the dividend.

## EXAMPLE 4 Selecting a strategy to determine the factors of a polynomial

$2x + 3$ is one factor of the function $f(x) = 6x^3 + 5x^2 - 16x - 15$. Determine the other factors. Then determine the zeros, and sketch a graph of the polynomial.

### Solution

$$\begin{aligned} (2x + 3) &= 2\left(x + \frac{3}{2}\right) \\ &= 2\left(x - \left(-\frac{3}{2}\right)\right) \rightarrow k = -\frac{3}{2} \end{aligned}$$

To use synthetic division, the divisor must be of the form $(x - k)$. Rewrite the divisor by dividing out the common factor 2 (the coefficient of $x$).

The division can now be done in two steps.

$$\begin{array}{r|rrrr} -\frac{3}{2} & 6 & 5 & -16 & -15 \\ & \downarrow & -9 & 6 & 15 \\ \hline & 6 & -4 & -10 & 0 \end{array}$$

First, divide $6x^3 + 5x^2 - 16x - 15$ by $\left(x + \frac{3}{2}\right)$. This means that

$$\frac{6x^3 + 5x^2 - 16x - 15}{\left(x + \frac{3}{2}\right)} = 6x^2 - 4x - 10 + \frac{0}{\left(x + \frac{3}{2}\right)}.$$

$$\frac{1}{2} \times \left[\frac{6x^3 + 5x^2 - 16x - 15}{\left(x + \frac{3}{2}\right)}\right]$$

$$= \frac{1}{2} \times \left[6x^2 - 4x - 10 + \frac{0}{\left(x + \frac{3}{2}\right)}\right]$$

$$\frac{6x^3 + 5x^2 - 16x - 15}{2\left(x + \frac{3}{2}\right)} = \frac{6x^2}{2} - \frac{4x}{2} - \frac{10}{2} + \frac{0}{2\left(x + \frac{3}{2}\right)}$$

$$\frac{6x^3 + 5x^2 - 16x - 15}{(2x + 3)} = 3x^2 - 2x - 5 + 0$$

Second, since the original divisor was $(2x + 3)$ or $2\left(x + \frac{3}{2}\right)$, multiply both sides by $\frac{1}{2}$ to get the correct multiplication statement.

Notice what this means—we only needed to divide our solution by 2 in the synthetic division.

$$\begin{array}{r|rrrr} -\frac{3}{2} & 6 & 5 & -16 & -15 \\ & \downarrow & -9 & 6 & 15 \\ \hline & 6 & -4 & -10 & 0 \\ & \div 2 & \div 2 & \div 2 & \div 2 \\ & 3 & -2 & -5 & 0 \end{array}$$

There is no remainder, which verifies that $2x + 3$ is a factor of the dividend.

$$\begin{aligned} f(x) &= 6x^3 + 5x^2 - 16x - 15 \\ &= (2x + 3)(3x^2 - 2x - 5) \\ &= (2x + 3)(3x - 5)(x + 1) \end{aligned}$$

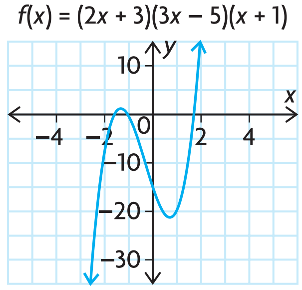
Factor the quotient.

Since $f(x) = (2x + 3)(3x - 5)(x + 1)$, the zeros are $-\frac{3}{2}$, $\frac{5}{3}$, and $-1$.

Determine the zeros by setting each factor equal to zero and solving for $x$.

An approximate graph of $y = f(x)$ is shown below.

$f(x) = (2x + 3)(3x - 5)(x + 1)$




Use the zeros to locate and plot the $x$-intercepts. Determine the $y$-intercept, and plot this point. Examine the standard and factored forms of the equation to determine the end behaviours of the function and the shape of the graph near the zeros. Sketch the graph.

## In Summary

### Key Idea

- Polynomials can be divided in much the same way that numbers are divided.

### Need to Know

- A polynomial can be divided by a polynomial of the same degree or less.
- Synthetic division is a shorter form of polynomial division. It can only be used when the divisor is linear (that is, $(x - k)$ or $(ax - k)$).
- When using polynomial or synthetic division,
  - terms should be arranged in descending order of degree, in both the divisor and the dividend, to make the division easier to perform
  - zero must be used as the coefficient of any missing powers of the variable in both the divisor and the dividend
- If the remainder of polynomial or synthetic division is zero, both the divisor and the quotient are factors of the dividend.

## CHECK *Your Understanding*

**1.** **a)** Divide $x^4 - 16x^3 + 4x^2 + 10x - 11$ by each of the following binomials.

**i)** $x - 2$ **ii)** $x + 4$ **iii)** $x - 1$

**b)** Are any of the binomials in part a) factors of $x^4 - 16x^3 + 4x^2 + 10x - 11$? Explain.

**2.** State the degree of the quotient for each of the following division statements, if possible.

**a)** $(x^4 - 15x^3 + 2x^2 + 12x - 10) \div (x^2 - 4)$

**b)** $(5x^3 - 4x^2 + 3x - 4) \div (x + 3)$

**c)** $(x^4 - 7x^3 + 2x^2 + 9x) \div (x^3 - x^2 + 2x + 1)$

**d)** $(2x^2 + 5x - 4) \div (x^4 + 3x^3 - 5x^2 + 4x - 2)$

**3.** Complete the divisions in question 2, if possible.

**4.** Complete the following table.

| Dividend | Divisor | Quotient | Remainder |
|---|---|---|---|
| $2x^3 - 5x^2 + 8x + 4$ | $x + 3$ | $2x^2 - 11x + 41$ | |
| | $2x + 4$ | $3x^3 - 5x + 8$ | $-3$ |
| $6x^4 + 2x^3 + 3x^2 - 11x - 9$ | | $2x^3 + x - 4$ | $-5$ |
| $3x^3 + x^2 - 6x + 16$ | $x + 2$ | | $8$ |

## PRACTISING

**5.** Calculate each of the following using long division.
K
**a)** $(x^3 - 2x + 1) \div (x - 4)$
**b)** $(x^3 + 2x^2 - 6x + 1) \div (x + 2)$
**c)** $(2x^3 + 5x^2 - 4x - 5) \div (2x + 1)$
**d)** $(x^4 + 3x^3 - 2x^2 + 5x - 1) \div (x^2 + 7)$
**e)** $(x^4 + 6x^2 - 8x + 12) \div (x^3 - x^2 - x + 1)$
**f)** $(x^5 + 4x^4 + 9x + 8) \div (x^4 + x^3 + x^2 + x - 2)$

**6.** Calculate each of the following using synthetic division.
**a)** $(x^3 - 7x - 6) \div (x - 3)$
**b)** $(2x^3 - 7x^2 - 7x + 19) \div (x - 1)$
**c)** $(6x^4 + 13x^3 - 34x^2 - 47x + 28) \div (x + 3)$
**d)** $(2x^3 + x^2 - 22x + 20) \div (2x - 3)$
**e)** $(12x^4 - 56x^3 + 59x^2 + 9x - 18) \div (2x + 1)$
**f)** $(6x^3 - 2x - 15x^2 + 5) \div (2x - 5)$

**7.** Each divisor was divided into another polynomial, resulting in the given quotient and remainder. Find the other polynomial (the dividend).
**a)** divisor: $x + 10$, quotient: $x^2 - 6x + 9$, remainder: $-1$
**b)** divisor: $3x - 2$, quotient: $x^3 + x - 12$, remainder: 15
**c)** divisor: $5x + 2$, quotient: $x^3 + 4x^2 - 5x + 6$, remainder: $x - 2$
**d)** divisor: $x^2 + 7x - 2$, quotient: $x^4 + x^3 - 11x + 4$, remainder: $x^2 - x + 5$

**8.** Determine the remainder, $r$, to make each multiplication statement true.
**a)** $(2x - 3)(3x + 5) + r = 6x^2 + x + 5$
**b)** $(x + 3)(x + 5) + r = x^2 + 9x - 7$
**c)** $(x + 3)(x^2 - 1) + r = x^3 + 3x^2 - x - 3$
**d)** $(x^2 + 1)(2x^3 - 1) + r = 2x^5 + 2x^3 + x^2 + 1$

**9.** Each dividend was divided by another polynomial, resulting in the given quotient and remainder. Find the other polynomial (the divisor).
**a)** dividend: $5x^3 + x^2 + 3$, quotient: $5x^2 - 14x + 42$, remainder: $-123$
**b)** dividend: $10x^4 - x^2 + 20x - 2$, quotient: $10x^3 - 100x^2 + 999x - 9970$, remainder: 99 698
**c)** dividend: $x^4 + x^3 - 10x^2 - 1$, quotient: $x^3 - 3x^2 + 2x - 8$, remainder: 31
**d)** dividend: $x^3 + x^2 + 7x - 7$, quotient: $x^2 + 3x + 13$, remainder: 19

**10.** Determine whether each binomial is a factor of the given polynomial.

**a)** $x + 5$, $x^3 + 6x^2 - x - 30$
**b)** $x + 2$, $x^4 - 5x^2 + 4$
**c)** $x - 2$, $x^4 - 5x^2 + 6$
**d)** $2x - 1$, $2x^4 - x^3 - 4x^2 + 2x + 1$
**e)** $3x + 5$, $3x^6 + 5x^5 + 9x^2 + 17x - 1$
**f)** $5x - 1$, $5x^4 - x^3 + 10x - 10$

**11.** **A** The volume of a rectangular box is $(x^3 + 6x^2 + 11x + 6)$ cm$^3$. The box is $(x + 3)$ cm long and $(x + 2)$ cm wide. How high is the box?

**12.** **a)** $8x^3 + 10x^2 - px - 5$ is divisible by $2x + 1$. There is no remainder. Find the value of $p$.
**b)** When $x^6 + x^4 - 2x^2 + k$ is divided by $1 + x^2$, the remainder is 5. Find the value of $k$.

**13.** The polynomial $x^3 + px^2 - x - 2$, $p \in \mathbf{R}$, has $x - 1$ as a factor. What is the value of $p$?

**14.** **T** Let $f(x) = x^n - 1$, where $n$ is an integer and $n \geq 1$. Is $f(x)$ always divisible by $x - 1$? Justify your decision.

**15.** If the divisor of a polynomial, $f(x)$, is $x - 4$, then the quotient is $x^2 + x - 6$ and the remainder is 7.

**a)** Write the division statement.
**b)** Rewrite the division statement by factoring the quotient.
**c)** Graph $f(x)$ using your results in part b).

**16.** **C** Use an example to show how synthetic division is essentially the same as regular polynomial division.

## Extending

**17.** The volume of a cylindrical can is $(4\pi x^3 + 28\pi x^2 + 65\pi x + 50\pi)$ cm$^3$. The can is $(x + 2)$ cm high. What is the radius?

**18.** Divide.

**a)** $(x^4 + x^3y - xy^3 - y^4) \div (x^2 - y^2)$
**b)** $(x^4 - 2x^3y + 2x^2y^2 - 2xy^3 + y^4) \div (x^2 + y^2)$

**19.** Is $x - y$ a factor of $x^3 - y^3$? Justify your answer.

**20.** If $f(x) = (x + 5)q(x) + (x + 3)$, what is the first multiple of $(x + 5)$ that is greater than $f(x)$?