3.6 Factoring Polynomials

GOAL

Make connections between a polynomial function and its remainder when divided by a binomial.

YOU WILL NEED

- graphing calculator

INVESTIGATE the Math

Consider the polynomial function $f(x) = x^3 + 4x^2 + x - 6$.

? How can you determine the factors of a polynomial function of degree 3 or greater?

A. For $f(x) = x^3 + 4x^2 + x - 6$, determine $f(2)$.

B. Determine the quotient of $\frac{f(x)}{x - 2}$, and state the remainder of the division. What do you notice?

C. Predict what the remainder of the division $\frac{f(x)}{x + 2}$ will be. What does this tell you about the relationship between $f(x)$ and $x + 2$?

D. Copy and complete the following table by choosing eight additional values of x. Use both positive and negative values. Leave space to add more columns in part E.

a	$f(a)$
2	20
−2	

E. Add the following two columns to your table, and complete your table for the other values of x.

a	$f(a)$	$\frac{f(x)}{x - a}$	Remainder
2	20	$\frac{f(x)}{x - 2} = x^2 + 6x + 13 + \frac{20}{x - 2}$	20

F. For which values of a in your table is $x - a$ a factor of $f(x)$? Can you see a pattern? Explain how you know there is a pattern.

G. How do the values of a that you identified in part F relate to the graph of $f(x)$?

H. Use your table and/or the graph to determine all the factors of $f(x)$.

I. Create a new factorable function, $g(x)$, and check whether the pattern you saw in part F exists for your new function.

Reflecting

J. What is the relationship between $f(a)$ and the quotient $\frac{f(x)}{x - a}$?

K. What is the value of $f(a)$ when $x - a$ is a factor?

L. How can you use your answer in part K to determine the factors of a polynomial?

EXAMPLE 1 Using reasoning to determine a remainder

Determine the remainder when $x^3 + 7x^2 + 2x - 5$ is divided by $x + 7$.

Solution

Let $f(x) = x^3 + 7x^2 + 2x - 5.$ ← Assign a function name to the expression given.

$f(x) = (x + 7)(\text{quotient}) + \text{remainder}$ ← $f(x)$ can be written as a division statement, with the divisor $x + 7$ multiplied by some quotient plus some remainder.

$$\begin{aligned} f(-7) &= (0)(\text{quotient}) + \text{remainder} \\ &= 0 + \text{remainder} \\ &= \text{remainder} \end{aligned}$$

← If $x = -7$, the divisor will be equal to 0 and the value of the function will be equal to the remainder.

$$\begin{aligned} f(-7) &= (-7)^3 + 7(-7)^2 + 2(-7) - 5 \\ &= -19 \end{aligned}$$

The remainder is -19.

remainder theorem

when a polynomial, $f(x)$, is divided by $x - a$, the remainder is equal to $f(a)$. If the remainder is zero, then $x - a$ is a factor of the polynomial. This can be used to help factor polynomials.

From Example 1, when $f(x)$ is divided by $x - 7$, the remainder is $f(7)$. This can be generalized into a theorem, known as the remainder theorem.

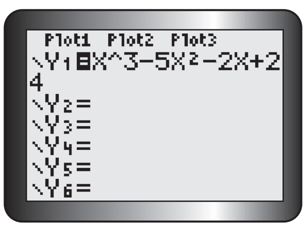
EXAMPLE 2 Selecting tools and strategies to factor a polynomial

Factor $x^3 - 5x^2 - 2x + 24$ completely.

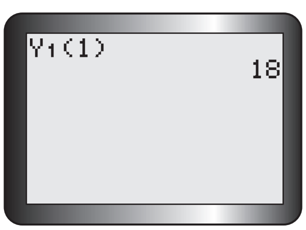
Solution

Let $f(x) = x^3 - 5x^2 - 2x + 24$.
Possible values of a: $\pm1, \pm2, \pm3, \pm4, \pm6, \pm8, \pm12, \pm24$
$x - a$ is a factor if $f(a) = 0$.

Factors of $f(x)$ will be of the form $x - a$, since the leading coefficient of $f(x)$ is 1. Since a must divide into the constant term, the possible values of a are the factors of 24.

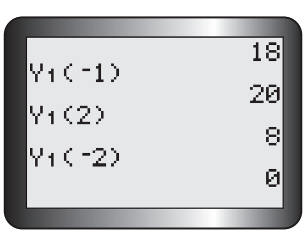


Using a graphing calculator makes this process much faster. Enter the equation into Y1.



In the home screen, enter Y1(1). This will give you the remainder when $f(x)$ is divided by $x - 1$.

$f(1) = 18$, so $x - 1$ is not a factor.



$f(-1) = 20$
$f(2) = 8$
$f(-2) = 0$

Therefore, $x + 2$ is a factor.

Repeat this process until you find a value of a that results in a remainder of zero. The factor will be of the form $x - a$.

-2	1	-5	-2	24
	↓	-2	14	-24
	1	-7	12	0

Use synthetic or regular polynomial division to divide $f(x)$ by $x + 2$.

$$f(x) = (x + 2)(x^2 - 7x + 12)$$
$$= (x + 2)(x - 4)(x - 3)$$

Factor the quotient.

factor theorem

$x - a$ is a factor of $f(x)$, if and only if $f(a) = 0$

Communication *Tip*

"A if and only if B" means that if A is true, then B is also true, and if B is true, then A is also true.

So "$x - a$ is a factor of $f(x)$, if and only if $f(a) = 0$" means that if $x - a$ is a factor of $f(x)$, then $f(a) = 0$, and if $f(a) = 0$, then $x - a$ is a factor of $f(x)$.

The **factor theorem** is a special case of the remainder theorem.

EXAMPLE 3 Connecting the factor theorem to characteristics of the graph of a polynomial function

Sketch a graph of the function $y = 4x^4 + 6x^3 - 6x^2 - 4x$.

Solution

$$\begin{aligned} y &= 4x^4 + 6x^3 - 6x^2 - 4x \\ &= 2x(2x^3 + 3x^2 - 3x - 2) \end{aligned}$$

First, divide out any common factors of the polynomial.

Let $f(x) = 2x^3 + 3x^2 - 3x - 2$.

$$\begin{aligned} f(1) &= 2(1)^3 + 3(1)^2 - 3(1) - 2 \\ &= 0 \end{aligned}$$

$x - 1$ is a factor.

Use the factor theorem to factor the remaining cubic.

1	2	3	−3	−2
	↓	2	5	2
	2	5	2	0

Divide to determine the other factors.

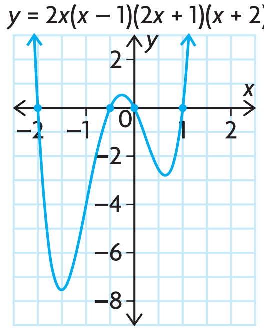
$$\begin{aligned} y &= 2x(x - 1)(2x^2 + 5x + 2) \\ &= 2x(x - 1)(2x + 1)(x + 2) \end{aligned}$$

Factor the quotient.

The function has zeros at $x = 0, 1, -\frac{1}{2}$, and -2.

State the zeros.

$y = 2x(x - 1)(2x + 1)(x + 2)$



Sketch a graph using the zeros and other characteristics from the standard and factored forms of the polynomial equations.

Since the degree is even and the leading coefficient is positive, the graph extends from the second quadrant to the first quadrant.

The function $y = 4x^4 + 6x^3 - 6x^2 - 4x + 0$ has a y-intercept of 0.

Each factor of $y = 2x(x - 1)(2x + 1)(x + 2)$ is order 1, so the graph has a linear shape near each zero.

EXAMPLE 4 Using a grouping strategy to factor polynomials

Factor $x^4 - 6x^3 + 2x^2 - 12x$.

Solution

$x^4 - 6x^3 + 2x^2 - 12x$

$= (x^4 - 6x^3) + (2x^2 - 12x)$ ← Group the first two terms and last two terms together.

$= x^3(x - 6) + 2x(x - 6)$ ← Divide out the common factors from each binominal.

$= (x - 6)(x^3 + 2x)$ ← Divide out the common factor of $x - 6$.

$= x(x - 6)(x^2 + 2)$ ← Divide out the common factor of x.

EXAMPLE 5 Connecting to prior knowledge to solve a problem

When $2x^3 - mx^2 + nx - 2$ is divided by $x + 1$, the remainder is -12 and $x - 2$ is a factor. Determine the values of m and n.

Solution

Let $f(x) = 2x^3 - mx^2 + nx - 2$.

$(x + 1) \rightarrow$ remainder -12	$(x - 2) \rightarrow$ remainder 0
$f(-1) = -12$	$f(2) = 0$
$2(-1)^3 - m(-1)^2 + n(-1) - 2 = -12$	$2(2)^3 - m(2)^2 + n(2) - 2 = 0$
$-2 - m - n - 2 = -12$	$16 - 4m + 2n - 2 = 0$
① $8 - n = m$	② $-4m + 2n = -14$

Set up two equations using the information given.

Simplify both equations.

Now you have a linear system of two equations in two unknowns.

Substitute equation $8 - n$ from equation ① into m in equation ②.

$-4(8 - n) + 2n = -14$ ← Solve this system of equations. Use substitution.

$-32 + 4n + 2n = -14$

$6n = 18$

$n = 3$

$8 - (3) = m$

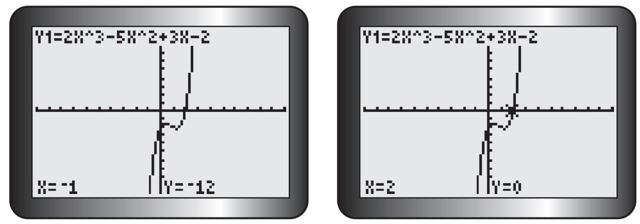
$5 = m$

Substitute $n = 3$ into ①.

$n = 3$ and $m = 5$

The original polynomial is $f(x) = 2x^3 - 5x^2 + 3x - 2$.

To check, verify that $f(-1) = -12$ and $f(2) = 0$.



In Summary

Key Ideas

- The remainder theorem: When a polynomial, $f(x)$, is divided by $x - a$, the remainder is equal to $f(a)$.
- The factor theorem: $x - a$ is a factor of $f(x)$, if and only if $f(a) = 0$.

Need to Know

- To factor a polynomial, $f(x)$, of degree 3 or greater,
  - use the Factor Theorem to determine a factor of $f(x)$
  - divide $f(x)$ by $x - a$
  - factor the quotient, if possible
- If a polynomial, $f(x)$, has a degree greater than 3, it may be necessary to use the factor theorem more than once.
- Not all polynomial functions are factorable.

CHECK Your Understanding

1. **a)** Given $f(x) = x^4 + 5x^3 + 3x^2 - 7x + 10$, determine the remainder when $f(x)$ is divided by each of the following binomials, without dividing.

i) $x - 2$

ii) $x + 4$

iii) $x - 1$

b) Are any of the binomials in part a) factors of $f(x)$? Explain.

2. Which of the following functions are divisible by $x - 1$?

a) $f(x) = x^4 - 15x^3 + 2x^2 + 12x - 10$

b) $g(x) = 5x^3 - 4x^2 + 3x - 4$

c) $h(x) = x^4 - 7x^3 + 2x^2 + 9x$

d) $j(x) = x^3 - 1$

3. Determine all the factors of the function $f(x) = x^3 + 2x^2 - 5x - 6$.

PRACTISING

4. State the remainder when $x + 2$ is divided into each polynomial.
K
   a) $x^2 + 7x + 9$
   b) $6x^3 + 19x^2 + 11x - 11$
   c) $x^4 - 5x^2 + 4$
   d) $x^4 - 2x^3 - 11x^2 + 10x - 2$
   e) $x^3 + 3x^2 - 10x + 6$
   f) $4x^4 + 12x^3 - 13x^2 - 33x + 18$

5. Determine whether $2x - 5$ is a factor of each polynomial.
   a) $2x^3 - 5x^2 - 2x + 5$
   b) $3x^3 + 2x^2 - 3x - 2$
   c) $2x^4 - 7x^3 - 13x^2 + 63x - 45$
   d) $6x^4 + x^3 - 7x^2 - x + 1$

6. Factor each polynomial using the factor theorem.
   a) $x^3 - 3x^2 - 10x + 24$
   b) $4x^3 + 12x^2 - x - 15$
   c) $x^4 + 8x^3 + 4x^2 - 48x$
   d) $4x^4 + 7x^3 - 80x^2 - 21x + 270$
   e) $x^5 - 5x^4 - 7x^3 + 29x^2 + 30x$
   f) $x^4 + 2x^3 - 23x^2 - 24x + 144$

7. Factor fully.
   a) $f(x) = x^3 + 9x^2 + 8x - 60$
   b) $f(x) = x^3 - 7x - 6$
   c) $f(x) = x^4 - 5x^2 + 4$
   d) $f(x) = x^4 + 3x^3 - 38x^2 + 24x + 64$
   e) $f(x) = x^3 - x^2 + x - 1$
   f) $f(x) = x^5 - x^4 + 2x^3 - 2x^2 + x - 1$

8. Use the factored form of $f(x)$ to sketch the graph of each function in question 7.

9. The polynomial $12x^3 + kx^2 - x - 6$ has $2x - 1$ as one of its factors. Determine the value of k.

10. When $ax^3 - x^2 + 2x + b$ is divided by $x - 1$, the remainder is 10. When it is divided by $x - 2$, the remainder is 51. Find a and b.
A

11. Determine a general rule to help decide whether $x - a$ and $x + a$ are factors of $x^n - a^n$ and $x^n + a^n$.
T

12. The function $f(x) = ax^3 - x^2 + bx - 24$ has three factors. Two of these factors are $x - 2$ and $x + 4$. Determine the values of a and b, and then determine the other factor.

13. Consider the function $f(x) = x^3 + 4x^2 + kx - 4$. The remainder from $f(x) \div (x + 2)$ is twice the remainder from $f(x) \div (x - 2)$. Determine the value of k.

14. Show that $x - a$ is a factor of $x^4 - a^4$.

15. Explain why the factor theorem works.
C

Extending

16. Use the factor theorem to prove that $x^2 - x - 2$ is a factor of $x^3 - 6x^2 + 3x + 10$.

17. Prove that $x + a$ is a factor of $(x + a)^5 + (x + c)^5 + (a - c)^5$.