3.7 Factoring a Sum or Difference of Cubes

GOAL

Factor the sum and difference of cubes.

LEARN ABOUT the Math

Megan has been completing her factoring homework, but she is stuck on the fourth question. She would prefer not to use the factor theorem, so she is hoping that there is a shortcut for factoring this type of polynomial.

$4x^2 - 9$	$16y^2 - 25$
$= (2x + 3)(2x - 3)$	$= (4y + 5)(4y - 5)$
$a^2 - 100$	$x^3 - 27$
$= (a + 10)(a - 10)$	$= ?$

? How can you factor a sum of cubes or a difference of cubes in one step?

EXAMPLE 1 Selecting a strategy to factor a sum or difference of cubes

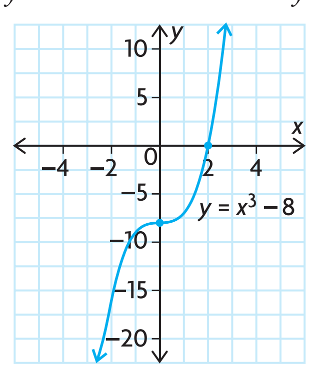
Factor the expressions $(ax)^3 - b^3$ and $(ax)^3 + b^3$ for your choice of values of a and b.

Solution A: Using a graph to factor a difference of cubes

Let $a = 1$ and $b = 2$.
Then, $(ax)^3 - b^3 = x^3 - 8$.

Substitute values of a and b.

$y = x^3 - 8$ is the same as $y = x^3$, translated 8 units down.



Use transformations to graph the function.

The graph of $y = x^3 - 8$ shows an x-intercept, which can be used to create one factor of the polynomial.

The only x-intercept is at 2, so $(x - 2)$ is one factor.

$$\begin{array}{r|rrrr} 2 & 1 & 0 & 0 & -8 \\ & \downarrow & 2 & 4 & 8 \\ \hline & 1 & 2 & 4 & 0 \end{array}$$

Divide $x^3 - 8$ by $x - 2$ to determine the other factor.

$x^3 - 8 = (x - 2)(x^2 + 2x + 4)$

If $x^2 + 2x + 4 = 0$, then the roots are $\frac{-2 \pm \sqrt{4 - 16}}{2}$.

$x^2 + 2x + 4$ cannot be factored further because the corresponding equation does not have real roots. The discriminant $b^2 - 4ac$ is equal to -12.

This is verified by the graph.

$x^3 - 8 = (x - 2)(x^2 + 2x + 4)$

Solution B: Using the factor theorem to factor a sum of cubes

Let $a = 2$ and $b = 3$.

Substitute values of a and b.

This gives the expression $f(x) = 8x^3 + 27$.

$f\left(-\frac{3}{2}\right) = 0$, so $2x + 3$ is a factor.

Use the factor theorem to determine one factor of $f(x)$.

$$\begin{array}{r|rrrr} -\frac{3}{2} & 8 & 0 & 0 & 27 \\ & \downarrow & -12 & 18 & -27 \\ \hline & 8 & -12 & 18 & 0 \end{array}$$

Divide to determine the other factors.

$$\begin{aligned} f(x) &= \left(x + \tfrac{3}{2}\right)(8x^2 - 12x + 18) \\ &= \left(x + \tfrac{3}{2}\right)(2)(4x^2 - 6x + 9) \\ f(x) &= (2x + 3)(4x^2 - 6x + 9) \end{aligned}$$

Multiplying $\left(x + \frac{3}{2}\right)$ by 2 results in the equivalent factor, $2x + 3$.

Solution C: Using a general solution

Let $a = 1$.

Substitute a value of 1 for a.

Then, $(ax)^3 - b^3 = x^3 - b^3$.

Let $f(x) = x^3 - b^3$.

$f(b) = 0$, so $(x - b)$ is a factor.

Use the factor theorem to determine one factor of $f(x)$.

$$\begin{array}{r|rrrr} b & 1 & 0 & 0 & -b^3 \\ & \downarrow & b & b^2 & b^3 \\ \hline & 1 & b & b^2 & 0 \end{array}$$

Divide to determine the other factors.

$f(x) = (x - b)(x^2 + bx + b^2)$

$x^3 - b^3 = (x - b)(x^2 + bx + b^2)$

If $b = 3$, $x^3 - 27 = (x - 3)(x^2 + 3x + 9)$.
If $b = 5$, $x^3 - 125 = (x - 5)(x^2 + 5x + 25)$.

Substitute different values of b.

Reflecting

A. Why would an expression such as $x^3 - 8$ be called a *difference of cubes?*

B. Why would an expression such as $8x^3 + 27$ be called a *sum of cubes?*

C. Why was the quadratic formula useful for determining that the second factor could not be factored further?

D. State a general factorization for the difference of cubes, $A^3 - B^3$, and for the sum of cubes, $A^3 + B^3$.

APPLY the Math

EXAMPLE 2 Selecting a strategy to factor a polynomial

Factor the expression $27x^3 + 125$.

Solution

$27x^3 + 125$
$= (3x)^3 + (5)^3$
This is a sum of cubes.

Any sum of cubes can be factored as follows:
$A^3 + B^3 = (A + B)(A^2 - AB + B^2)$

$= (3x + 5)(9x^2 - 15x + 25)$

Use this factorization to write the two factors, if $A = 3x$ and $B = 5$.

EXAMPLE 3 Connecting prior knowledge to factor a polynomial

Factor $7x^4 - 448x$.

Solution

$7x^4 - 448x$

$= 7x(x^3 - 64)$

Divide out the common factor. This leaves a difference of cubes. Any difference of cubes can be factored as follows: $A^3 - B^3 = (A - B)(A^2 + AB + B^2)$

$= 7x(x - 4)(x^2 + 4x + 16)$

Use this factorization to write the factors, if $A = x$ and $B = 4$.

EXAMPLE 4 Selecting a strategy to factor a polynomial that is not obviously a cubic

Factor the expression $x^9 - 512$ completely.

Solution

$x^9 - 512$

$= (x^3)^3 - (8)^3$

Write the expression as the difference of two cubes.

$= (x^3 - 8)(x^6 + 8x^3 + 64)$

Use the factorization $A^3 - B^3 = (A - B)(A^2 + AB + B^2)$ to factor the expression, if $A = x^3$ and $B = 8$.

$= (x - 2)(x^2 + 2x + 4)(x^6 + 8x^3 + 64)$

$(x^3 - 8)$ is also a difference of cubes, so factor it further using the pattern where $A = x$ and $B = 2$.

In Summary

Key Ideas

- An expression that contains two perfect cubes that are added together is called a sum of cubes and can be factored as follows:

  $A^3 + B^3 = (A + B)(A^2 - AB + B^2)$

- An expression that contains perfect cubes where one is subtracted from the other is called a difference of cubes and can be factored as follows:

  $A^3 - B^3 = (A - B)(A^2 + AB + B^2)$

CHECK *Your Understanding*

1. Using Solution C for Example 1 as a model, determine the factors of $x^3 + b^3$.

2. Factor each of the following expressions.

a) $x^3 - 64$	**d)** $8x^3 - 27$	**g)** $27x^3 + 8$
b) $x^3 - 125$	**e)** $64x^3 - 125$	**h)** $1000x^3 + 729$
c) $x^3 + 8$	**f)** $x^3 + 1$	**i)** $216x^3 - 8$

3. Factor each expression.

a) $64x^3 + 27y^3$	**c)** $(x + 5)^3 - (2x + 1)^3$
b) $-3x^4 + 24x$	**d)** $x^6 + 64$

PRACTISING

4. Factor.

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a) $x^3 - 343$	**d)** $125x^3 - 512$	**g)** $512x^3 + 1$
b) $216x^3 - 1$	**e)** $64x^3 - 1331$	**h)** $1331x^3 + 1728$
c) $x^3 + 1000$	**f)** $343x^3 + 27$	**i)** $512 - 1331x^3$

5. Factor each expression.

a) $\frac{1}{27}x^3 - \frac{8}{125}$	**c)** $(x - 3)^3 + (3x - 2)^3$
b) $-432x^5 - 128x^2$	**d)** $\frac{1}{512}x^9 - 512$

6. Jarred claims that the expression

A $\frac{(a + b)(a^2 - ab + b^2) + (a - b)(a^2 + ab + b^2)}{2a^3}$ is equivalent to 1.

Do you agree or disagree with Jarred? Justify your decision.

7. 1729 is a very interesting number. It is the smallest whole number that can be expressed as a sum of two cubes in two ways: $1^3 + 12^3$ and $9^3 + 10^3$. Use the factorization for the sum of cubes to verify that these sums are equal.

8. Prove that $(x^2 + y^2)(x^4 - x^2y^2 + y^4)(x^{12} - x^6y^6 + y^{12}) + 2x^9y^9$

T equals $(x^9 + y^9)^2$ using the factorization for the sum of cubes.

9. Some students might argue that if you know how to factor a sum

C of cubes, then you do not need to know how to factor a difference of cubes. Explain why you agree or disagree.

Extending

10. The number 1729, in question 7, is called a taxicab number.

a) Use the Internet to find out why 1729 is called a taxicab number.

b) Are there other taxicab numbers? If so, what are they?