4.1 Solving Polynomial Equations

YOU WILL NEED

- graphing calculator or graphing software

GOAL

Solve polynomial equations using a variety of strategies.

LEARN ABOUT the Math

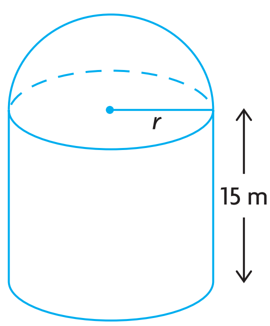
Amelia's family is planning to build another silo for grain storage, identical to those they have on their farm. The cylindrical portion of those they currently have is 15 m tall, and the silo's total volume is 684π m^3.

? What are the possible values for the radius of the new silo?

EXAMPLE 1 Representing a problem with a polynomial model

Determine possible values for the radius of the silo.

Solution



Draw a diagram to represent the silo. In this case, the height must be 15 m.

$$V = V_{cylinder} + V_{hemisphere}$$
$$V = \pi r^2 h + \frac{1}{2}\left(\frac{4}{3}\pi r^3\right)$$

Determine a **polynomial equation** for the volume of the silo using the formula for the volume of a cylinder and a hemisphere.

polynomial equation
an equation in which one polynomial expression is set equal to another (e.g., $x^3 - 5x^2 = 4x - 3$, or $5x^4 - 3x^3 + x^2 - 6x = 9$)

$$684\pi = \pi r^2(15) + \frac{2}{3}\pi r^3$$
$$684\pi = 15\pi r^2 + \frac{2}{3}\pi r^3$$

Substitute the given values for the volume and the height into the formula and simplify the equation.

$$0 = 15\pi r^2 + \frac{2}{3}\pi r^3 - 684\pi$$
$$0 = \frac{\pi}{3}(45r^2 + 2r^3 - 2052)$$

Divide out the common factor of $\frac{\pi}{3}$, then divide both sides of the equation by this value.

$$0 = 45r^2 + 2r^3 - 2052$$
$$0 = 2r^3 + 45r^2 - 2052$$

Rewrite the polynomial in descending order of degree.

$$\text{Let } f(r) = 2r^3 + 45r^2 - 2052$$

Solve the equation by factoring the polynomial.

$$f(2) = 2(2)^3 + 45(2)^2 - 2052$$
$$= -1856$$
$$f(3) = 2(3)^3 + 45(3)^2 - 2052$$
$$= -1593$$
$$f(6) = 2(6)^3 + 45(6)^2 - 2052$$
$$= 0$$

Since the roots of the equation are the x-intercepts of the related function, use the factor theorem to determine one factor.

By the factor theorem, $(r - 6)$ is a factor of $f(r)$.

$$(2r^3 + 45r^2 - 2052) \div (r - 6)$$

$$\begin{array}{r|rrrr} 6 & 2 & 45 & 0 & -2052 \\ & \downarrow & 12 & 342 & 2052 \\ \hline & 2 & 57 & 342 & 0 \end{array}$$

Use synthetic division to divide $f(r)$ by $(r - 6)$ to determine the other factor.

$$(r - 6)(2r^2 + 57r + 342) = 0$$

$r - 6 = 0$

$r = 6$ is one solution

Set each factor equal to zero and solve.

$$2r^2 + 57r + 342 = 0$$

$$a = 2, b = 57, c = 342$$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$r = \frac{-57 \pm \sqrt{(57)^2 - 4(2)(342)}}{2(2)}$$

$$r = \frac{-57 \pm \sqrt{513}}{4}$$

$$r = \frac{-57 + \sqrt{513}}{4} \quad \text{or} \quad r = \frac{-57 - \sqrt{513}}{4}$$

$$r \doteq -8.6 \qquad\qquad r \doteq -19.9$$

The second factor does not appear to be factorable, so use the quadratic formula to solve for the other roots.

The silo must have a radius of 6 m.

The radius cannot be negative, and so only the positive root can be the radius of the silo.

Reflecting

A. How could you verify the solutions you found, with and without using a graphing calculator?

B. What restriction was placed on the variable in the polynomial equation? Explain why this was necessary.

C. Do you think it is possible to solve all cubic and quartic equations using an algebraic strategy involving factoring? Explain.

APPLY the Math

EXAMPLE 2 Selecting a strategy to solve a cubic equation

Solve $4x^3 - 12x^2 - x + 3 = 0$.

Solution A: Using the factor theorem

Let $f(x) = 4x^3 - 12x^2 - x + 3$

Possible values for x where $f(x) = 0$:

$\pm 1, \pm 3, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{1}{4}, \pm\frac{3}{4}$

Use the factor theorem and the related polynomial function to determine one factor of the equation. Numbers that could make $f(x) = 0$ are of the form $\frac{p}{q}$, where p is a factor of the constant term 3 and q is a factor of the leading coefficient 4.

$$\begin{aligned} f(1) &= 4(1)^3 - 12(1)^2 - (1) + 3 \\ &= -6 \end{aligned}$$

$$\begin{aligned} f(-1) &= 4(-1)^3 - 12(-1)^2 - (-1) + 3 \\ &= -12 \end{aligned}$$

$$\begin{aligned} f(3) &= 4(3)^3 - 12(3)^2 - (3) + 3 \\ &= 0 \end{aligned}$$

By the factor theorem, $(x - 3)$ is a factor of $f(x)$.

$(4x^3 - 12x^2 - x + 3) \div (x - 3)$

$$\begin{array}{r} 4x^2 + 0x - 1 \\ x - 3\overline{)4x^3 - 12x^2 - x + 3} \\ \underline{4x^3 - 12x^2} \\ 0x^2 - x \\ \underline{0x^2 - 0x} \\ -x + 3 \\ \underline{-x + 3} \\ 0 \end{array}$$

Divide $f(x)$ by $(x - 3)$ to find the second factor using either long or synthetic division.

$(x - 3)(4x^2 - 1) = 0$

The quotient $4x^2 - 1$ is a difference of squares. Factor this.

$(x - 3)(2x - 1)(2x + 1) = 0$

$x - 3 = 0$ or $2x - 1 = 0$ or $2x + 1 = 0$

$x = 3 \qquad 2x = 1 \qquad 2x = -1$

$x = \frac{1}{2} \qquad x = -\frac{1}{2}$

Set each of the factors equal to zero to solve.

Check:

$x = \frac{1}{2}$

LS	RS
$4x^3 - 12x^2 - x + 3$	0
$= 4\left(\frac{1}{2}\right)^3 - 12\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right) + 3$	
$= \frac{1}{2} - 3 - \frac{1}{2} + 3$	
$= 0$	
LS = RS ✓	

$x = -\frac{1}{2}$

LS	RS
$4x^3 - 12x^2 - x + 3$	0
$= 4\left(-\frac{1}{2}\right)^3 - 12\left(-\frac{1}{2}\right)^2 - \left(-\frac{1}{2}\right) + 3$	
$= -\frac{1}{2} - 3 + \frac{1}{2} + 3$	
$= 0$	
LS = RS ✓	

Verify the solutions by substitution.

You only need to check $x = \frac{1}{2}$ and $-\frac{1}{2}$ since $x = 3$ was obtained using substitution.

The solutions to $4x^3 - 12x^2 - x + 3 = 0$ are $x = -\frac{1}{2}$, $x = \frac{1}{2}$, and $x = 3$.

Solution B: Factoring by grouping

$4x^3 - 12x^2 - x + 3 = 0$

$4x^2(x - 3) - 1(x - 3) = 0$

$(x - 3)(4x^2 - 1) = 0$

$(x - 3)(2x - 1)(2x + 1) = 0$

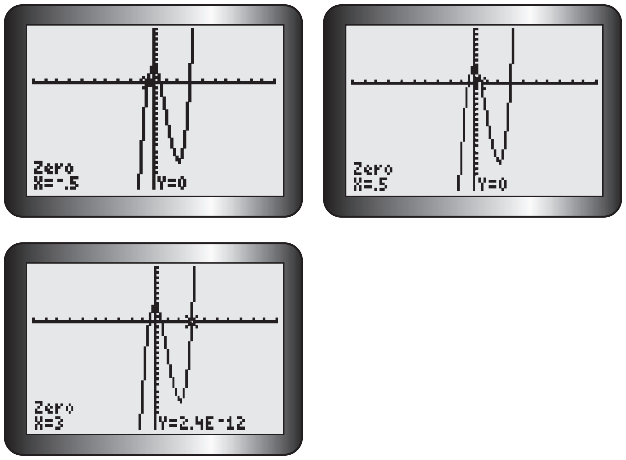
The first two terms and the last two terms have a common factor, so you can factor by grouping.

$x - 3 = 0$ or $2x - 1 = 0$ or $2x + 1 = 0$

$x = 3 \qquad 2x = 1$ or $\qquad 2x = -1$

$x = \frac{1}{2} \qquad x = -\frac{1}{2}$

Set each of the factors equal to zero to solve.



Verify the solutions, by graphing the corresponding polynomial function and determining its zeros.

The solutions to $4x^3 - 12x^2 - x + 3 = 0$ are $x = -\frac{1}{2}$, $x = \frac{1}{2}$, and $x = 3$.

EXAMPLE 3 Selecting tools to solve a question involving modelling

The paths of two orcas playing in the ocean were recorded by some oceanographers. The first orca's path could be modelled by the equation $h(t) = 2t^4 - 17t^3 + 27t^2 - 252t + 232$, and the second by $h(t) = 20t^3 - 200t^2 + 300t - 200$, where h is their height above/below the water's surface in centimetres and t is the time during the first 8 s of play. Over this 8-second period, at what times were the two orcas at the same height or depth?

Solution

$$2t^4 - 17t^3 + 27t^2 - 252t + 232 = 20t^3 - 200t^2 + 300t - 200$$
$$2t^4 - 37t^3 + 227t^2 - 552t + 432 = 0$$

Since you are solving for the time when the heights or depths are the same, set the two equations equal to each other and use inverse operations to make the right side of the equation equal to zero.

Let $f(t) = 2t^4 - 37t^3 + 227t^2 - 552t + 432$

Solve the equation by factoring.

Some possible values for t where $f(t) = 0$:

$\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 9$

$f(1) = 72$

$f(2) = -28$

$f(3) = -18$

$f(4) = 0$

Use the factor theorem to determine one factor of $f(t)$. Since the question specifies that the time is within the first 10 s, you only need to consider values of $\frac{p}{q}$ between 0 and 10. In this case, consider just the factors of 432.

By the factor theorem, $(t - 4)$ is a factor of $f(t)$.

$(2t^4 - 37t^3 + 227t^2 - 552t + 432) \div (t - 4)$

$$\begin{array}{r|rrrrr} 4 & 2 & -37 & 227 & -552 & 432 \\ & \downarrow & 8 & -116 & 444 & -432 \\ \hline & 2 & -29 & 111 & -108 & 0 \end{array}$$

Divide $f(t)$ by $(t - 4)$ to determine the second factor.

$f(t) = (t - 4)(2t^3 - 29t^2 + 111t - 108)$

$f(6) = -108$

$f(8) = -208$

$f(9) = 0$

Since the second factor is cubic, you must continue looking for more zeros using the factor theorem.
It is not necessary to recheck the values that were used in an earlier step, so carry on with the other possibilities.

By the factor theorem, $(t - 9)$ is a factor of $f(t)$.

$(2t^3 - 29t^2 + 111t - 108) \div (t - 9)$

$$\begin{array}{r|rrrr} 9 & 2 & -29 & 111 & -108 \\ & \downarrow & 18 & -99 & 108 \\ \hline & 2 & -11 & 12 & 0 \end{array}$$

Divide the cubic polynomial by $(t - 9)$ to determine the other factor.

$$\begin{aligned} f(t) &= (t - 4)(t - 9)(2t^2 - 11t + 12) \\ &= (t - 4)(t - 9)(2t - 3)(t - 4) \\ &= (t - 4)^2(t - 9)(2t - 3) \end{aligned}$$

Factor the quadratic.

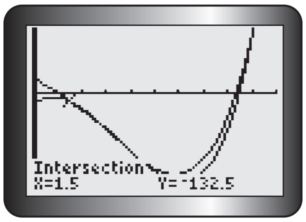
$(t - 4)^2 = 0$, $t - 9 = 0$, or $2t - 3 = 0$

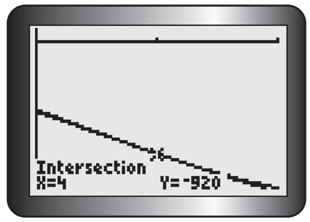
$t = 4 \qquad t = 9 \qquad t = 1.5$

Set each factor equal to zero and solve.

The solutions that are valid on the given domain are $t = 1.5$ and $t = 4$.

The polynomial functions given only model the orca's movement between 0 s and 8 s, so the solution $t = 9$ is inadmissable.





Graph both functions and adjust the window settings to determine the points of intersection to verify the solutions.

The orcas were at the same depth after 1.5 s and 4 s on the interval between 0 s and 8 s.

EXAMPLE 4 Selecting a strategy to solve a polynomial equation that is unfactorable

Solve each of the following.

a) $x^4 + 5x^2 = -1$ **b)** $x^3 - 2x + 3 = 0$

Solution

a) $x^4 + 5x^2 = -1$

$x^4 + 5x^2 + 1 = 0$

Add 1 to both sides of the equation to make the right side of the equation equal to zero.

Let $f(x) = x^4 + 5x^2 + 1$

$f(1) = (1)^4 + 5(1)^2 + 1 = 7$

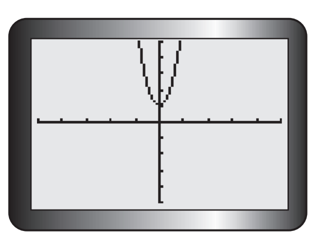
$f(-1) = (-1)^4 + 5(-1)^2 + 1 = 7$

If the equation is factorable, then either $f(1)$ or $f(-1)$ should give a value of 0.

The polynomial in this equation cannot be factored.

The function $f(x) = x^4 + 5x^2 + 1$ has an even degree and a positive leading coefficient, so its end behaviours are the same. In this case, as $x \to \pm\infty$, $y \to \infty$. This function has a degree of 4, so it could have 4, 3, 2, 1, or 0 x-intercepts.

Use the corresponding polynomial function to visualize the graph and determine the possible number of zeros.



Graph the function on a graphing calculator to determine its zeros.

Based on the graph and the function's end behaviours, it never crosses the x-axis, so it has no zeros. As a result, the equation $x^4 + 5x = -1$ has no solutions.

b) $x^3 - 2x + 3 = 0$

Let $f(x) = x^3 - 2x + 3$

$f(1) = (1)^3 - 2(1) + 3 = 2$

$f(-1) = (-1)^3 - 2(-1) + 3 = 4$

$f(3) = (3)^3 - 2(3) + 3 = 24$

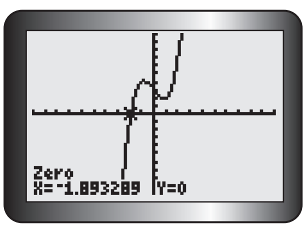
$f(-3) = (-3)^3 - 2(-3) + 3 = -18$

If the equation is factorable, then either $f(1)$, $f(-1)$, $f(3)$, or $f(-3)$ should equal 0.

The polynomial in this equation cannot be factored.

The function $f(x) = x^3 - 2x^2 + 3$ has an odd degree and a positive leading coefficient, so its end behaviours are opposite. In this case, as $x \rightarrow \infty$, $y \rightarrow \infty$, and as $x \rightarrow -\infty$, $y \rightarrow -\infty$. This function has a degree of 3, so it could have 3, 2, or 1 x-intercepts.

Use the corresponding polynomial function to visualize the graph and determine the possible number of zeros.



Graph the function on a graphing calculator to determine its zeros.

Based on the graph and the function's end behaviours, it crosses the x-axis only once. The solution is $x \doteq -1.89$.

In Summary

Key Idea

- The solutions to a polynomial equation $f(x) = 0$ are the zeros of the corresponding polynomial function, $y = f(x)$.

Need to Know

- Polynomial equations can be solved using a variety of strategies:
  - algebraically using a factoring strategy
  - graphically using a table of values, transformations, or a graphing calculator
- Only some polynomial equations can be solved by factoring, since not all polynomials are factorable. In these cases, graphing technology must be used.
- When solving problems using polynomial models, it may be necessary to ignore the solutions that are outside the domain defined by the conditions of the problem.

CHECK Your Understanding

1. State the zeros of the following functions.

a) $y = 2x(x - 1)(x + 2)(x - 2)$
b) $y = 5(2x + 3)(4x - 5)(x + 7)$
c) $y = 2(x - 3)^2(x + 5)(x - 4)$
d) $y = (x + 6)^3(2x - 5)$
e) $y = -5x(x^2 - 9)$
f) $y = (x + 5)(x^2 - 4x - 12)$

2. Solve each of the following equations by factoring. Verify your solutions using graphing technology.

a) $3x^3 = 27x$
b) $4x^4 = 24x^2 + 108$
c) $3x^4 + 5x^3 - 12x^2 - 20x = 0$
d) $10x^3 + 26x^2 - 12x = 0$
e) $2x^3 + 162 = 0$
f) $2x^4 = 48x^2$

3. a) Determine the zeros of the function $y = 2x^3 - 17x^2 + 23x + 42$.
b) Write the polynomial equation whose roots are the zeros of the function in part a).

4. Explain how you can solve
$x^3 + 12x^2 + 21x - 4 = x^4 - 2x^3 - 13x^2 - 4$
using two different strategies.

5. Determine the zeros of the function
$f(x) = 2x^4 - 11x^3 - 37x^2 + 156x$ algebraically.
Verify your solution using graphing technology.

PRACTISING

6. State the zeros of the following functions.

a) $f(x) = x(x - 2)^2(x + 5)$
b) $f(x) = (x^3 + 1)(x - 17)$
c) $f(x) = (x^2 + 36)(8x - 16)$
d) $f(x) = -3x^3(2x + 4)(x^2 - 25)$
e) $f(x) = (x^2 - x - 12)(3x)$
f) $f(x) = (x + 1)(x^2 + 2x + 1)$

7. Determine the roots algebraically by factoring.

a) $x^3 - 8x^2 - 3x + 90 = 0$
b) $x^4 + 9x^3 + 21x^2 - x - 30 = 0$
c) $2x^3 - 5x^2 - 4x + 3 = 0$
d) $2x^3 + 3x^2 = 5x + 6$
e) $4x^4 - 4x^3 - 51x^2 + 106x = 40$
f) $12x^3 - 44x^2 = -49x + 15$

8. Use graphing technology to find the real roots to two decimal places.

a) $x^3 - 7x + 6 = 0$

b) $x^4 - 5x^3 - 17x^2 + 3x + 18 = 0$

c) $3x^3 - 2x^2 + 16 = x^4 + 16x$

d) $x^5 + x^4 = 5x^3 - x^2 + 6x$

e) $105x^3 = 344x^2 - 69x - 378$

f) $21x^3 - 58x^2 + 10 = -18x^4 - 51x$

9. Solve each of the following equations.

K

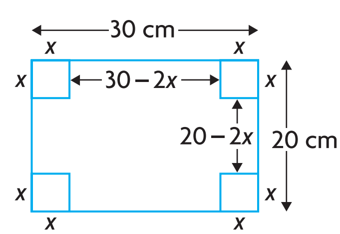
a) $x^3 - 6x^2 - x + 30 = 0$

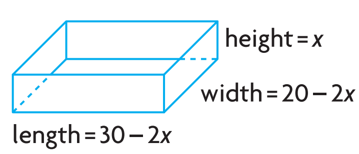
b) $9x^4 - 42x^3 + 64x^2 - 32x = 0$

c) $6x^4 - 13x^3 - 29x^2 + 52x = -20$

d) $x^4 - 6x^3 + 10x^2 - 2x = x^2 - 2x$

10. An open-topped box can be created by cutting congruent squares from each of the four corners of a piece of cardboard that has dimensions of 20 cm by 30 cm and folding up the sides. Determine the dimensions of the squares that must be cut to create a box with a volume of 1008 cm^3.





11. The Sickle-Lichti family members are very competitive card players.

A

They keep score using a complicated system that incorporates positives and negatives. Maya's score for the last game night could be modelled by the function $S(x) = x(x - 4)(x - 6)$, $x < 10$, $x \in \mathbf{W}$, where x represents the game number.

a) After which game was Maya's score equal to zero?

b) After which game was Maya's score -5?

c) After which game was Maya's score 16?

d) Draw a sketch of the graph of $S(x)$ if $x \in \mathbf{R}$. Explain why this graph is *not* a good model to represent Maya's score during this game night.

12. The function $s(t) = -\frac{1}{2}gt^2 + v_0t + s_0$ can be used to calculate s, the height above a planet's surface in metres, where g is the acceleration due to gravity, t is the time in seconds, v_0 is the initial velocity in metres per second, and s_0 is the initial height in metres. The acceleration due to gravity on Mars is $g = -3.92$ m/s^2. Find, to two decimal places, how long it takes an object to hit the surface of Mars if the object is dropped from 1000 m above the surface.

13. The distance of a ship from its harbour is modelled by the function $d(t) = -3t^3 + 3t^2 + 18t$, where t is the time elapsed in hours since departure from the harbour.
    a) Factor the time function.
    b) When does the ship return to the harbour?
    c) There is another zero of $d(t)$. What is it, and why is it not relevant to the problem?
    d) Draw a sketch of the function where $0 \leq t \leq 3$.
    e) Estimate the time that the ship begins its return trip back to the harbour.

14. During a normal 5 s respiratory cycle in which a person inhales and then exhales, the volume of air in a person's lungs can be modelled by $V(t) = 0.027t^3 - 0.27t^2 + 0.675t$, where the volume, V, is measured in litres at t seconds.
    a) What restriction(s) must be placed on t?
    b) If asked, "How many seconds have passed if the volume of air in a person's lungs is 0.25 L?" would you answer this question algebraically or by using graphing technology? Justify your decision.
    c) Solve the problem in part b).

15. Explain why the following polynomial equation has no real solutions:
$0 = 5x^8 + 10x^6 + 7x^4 + 18x^2 + 132$

16. **T** Determine algebraically where the cubic polynomal function that has zeros at 2, 3, and -5 and passes through the point (4, 36) has a value of 120.

17. **C** For each strategy below, create a cubic or quartic equation you might solve by using that strategy (the same equation could be used more than once). Explain why you picked the equation you did.
    a) factor theorem
    b) common factor
    c) factor by grouping
    d) quadratic formula
    e) difference or sum of cubes
    f) graphing technology

Extending

18. a) It is possible that a polynomial equation of degree 4 can have no real roots. Create such a polynomial equation and explain why it cannot have any real roots.
    b) Explain why a degree 5 polynomial equation must have at least one real root.

19. The factor theorem only deals with rational zeros. Create a polynomial of degree 5 that has no rational zeros. Explain why your polynomial has no rational zero but has at least one irrational zero.