4.2 Solving Linear Inequalities

GOAL

Solve linear inequalities.

YOU WILL NEED

- graphing calculator or graphing software

LEARN ABOUT the Math

In mathematics, you must be able to represent intervals and identify smaller sections of a relation or a set of numbers. You have used the following inequality symbols:

$>$ greater than $\quad$ $<$ less than

$\geq$ greater than or equal to $\quad$ $\leq$ less than or equal to

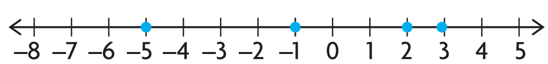
When you write one of these symbols between two or more linear expressions, the result is called a **linear inequality**.

linear inequality

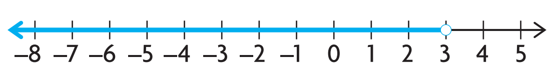
an inequality that contains an algebraic expression of degree 1 (e.g., $5x + 3 > 6x - 2$)

To solve an inequality, you have to find all the possible values of the variable that satisfy the inequality.

For example, $x = 2$ satisfies $3x - 1 < 8$, but so do $x = 2.9$, $x = -1$, and $x = -5$.



In fact, every real number less than 3 results in a number smaller than 8. So all real numbers less than 3 satisfy this inequality. The thicker part of the number line below represents this solution.



Communication *Tip*

To show that a number is not included in the solution set, use an open dot at this value. A solid dot shows that this value is included in the solution set.

The solution to $3x - 1 < 8$ can be written in set notation as $\{x \in \mathbf{R} \mid x < 3\}$ or in interval notation $x \in (-\infty, 3)$.

? How can you determine algebraically the solution set to a linear inequality like $3x - 1 < 8$?

EXAMPLE 1 Selecting an inverse operation strategy to solve a linear inequality

Solve the linear inequality $3x - 1 < 8$.

Solution

$$3x - 1 < 8$$

$$3x - 1 + 1 < 8 + 1$$

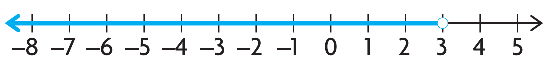
Treat the inequality like a linear equation and use inverse operations to isolate *x*. Add 1 to both sides of the inequality and simplify.

$$3x < 9$$

$$\frac{3x}{3} < \frac{9}{3}$$

Divide both sides of the inequality by 3.

$$x < 3$$



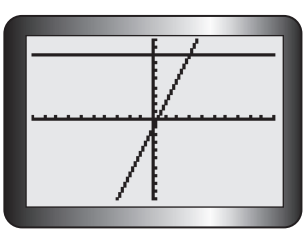
A number line helps visualize the solution.

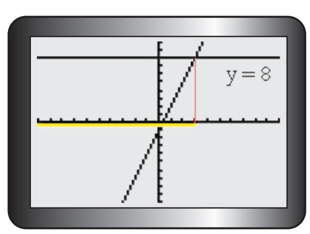
Check $x = 0$.

LS	RS
$3x - 1$	8
$= 3(0) - 1$	
$= -1$	

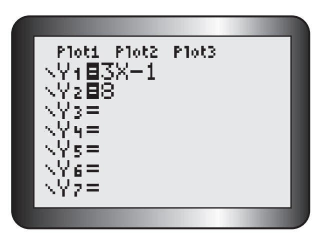
LS < RS

Choose a value for *x* that is less than 3 to verify that any number in the solution set satisfies the original inequality.





You can also verify the solution set using a graphing calculator. Graph each side of the inequality as a function.



The *y*-values on the line $y = 3x - 1$ that are less than 8 are found on all points that lie on the line below the horizontal line $y = 8$. This happens when *x* is smaller than 3.

The solution set is $\{x \in \mathbf{R} \mid x < 3\}$ or in interval notation $x \in (-\infty, 3)$.

Reflecting

A. How was solving a linear inequality like solving a linear equation? How was it different?

B. When checking the solution to an inequality, why is it not necessary for the left side to equal the right side?

C. Why do most linear equations have only one solution, but linear inequalities have many?

APPLY the Math

EXAMPLE 2 Using reasoning to determine which operations preserve the truth of a linear inequality

Can you add, subtract, multiply, or divide both sides of an inequality by a non-zero value and still have a valid inequality?

Solution

$$4 < 8$$

$$4 + 5 < 8 + 5$$

$$9 < 13$$

Write a true inequality using two numbers. Add the same positive quantity to both sides.

The result is still true.

$$4 + (-5) < 8 + (-5)$$

$$-1 < 3$$

Add the same negative quantity to both sides of the initial inequality.

The result is still true.

$$4 - 10 < 8 - 10$$

$$-6 < -2$$

Subtract the same positive quantity from both sides of the initial inequality.

The result is still true.

$$4 - (-3) < 8 - (-3)$$

$$7 < 11$$

Subtract the same negative quantity from both sides of the initial inequality.

The result is still true.

$$4(6) < 8(6)$$

$$24 < 48$$

Multiply by the same positive quantity on both sides of the initial inequality.

The result is still true.

$$4(-2) < 8(-2)$$
$$-8 < -16$$

Multiply by the same negative quantity on both sides of the initial inequality.

The result is false.
In this case, $-8 > -16$.

$$4 \div 2 < 8 \div 2$$
$$2 < 4$$

Divide by the same positive quantity on both sides of the initial inequality.

The result is still true.

$$4 \div (-2) < 8 \div (-2)$$
$$-2 < -4$$

Divide by the same negative quantity on both sides of the initial inequality.

The result is false.
In this case, $-2 > -4$.

Most of the operations preserve the validity of the inequality. The exception occurs when both sides are multiplied or divided by a negative number. In these two cases, reversing the inequality sign preserves the validity.

Since algebraic expressions represent numbers, this conclusion applies to linear inequalities that contain variables.

EXAMPLE 3 Reflecting to verify a solution

Solve the inequality $35 - 2x \geq 20$.

Solution

$$35 - 2x \geq 20$$
$$-2x \geq 20 - 35$$
$$-2x \geq -15$$

Use inverse operations to isolate *x*. Subtract 35 from both sides and simplify.

$$\frac{-2x}{-2} \leq \frac{-15}{-2}$$
$$x \leq 7.5$$

Divide both sides by -2. Since the division involves a negative number, reverse the inequality sign.

Represent the solution on a number line.
A solid dot is placed on 7.5 since this number is included in the solution set.

If $x = 5$, then

LS	RS
$35 - 2x$	20
$= 35 - 2(5)$	
$= 35 - 10$	
$= 25$	

LS $\geq$ RS: This is the desired outcome.

Test a value less than 7.5 and a value greater than 7.5 to verify the solution. Since $x = 5$ makes the inequality true and $x = 8$ makes the inequality false, the solution is correct.

If $x = 8$, then

LS	RS
$35 - 2x$	20
$= 35 - 2(8)$	
$= 35 - 16$	
$= 19$	

LS $\leq$ RS: This is not the desired outcome.

The solution set is $\{x \in \mathbf{R} | x \leq 7.5\}$ or in interval notation $x \in (-\infty, 7.5]$.

The value of 7.5 makes both sides equal. Since the inequality sign includes an equal sign, 7.5 must be part of the solution set.

EXAMPLE 4 Connecting the process of solving a double inequality to solving a linear inequality

Solve the inequality $30 \leq 3(2x + 4) - 2(x + 1) \leq 46$.

Solution

$30 \leq 3(2x + 4) - 2(x + 1) \leq 46$

This is a combination of two inequalities:
$30 \leq 3(2x + 4) - 2(x + 1)$ and $3(2x + 4) - 2(x + 1) \leq 46$.
A valid solution must satisfy both inequalities.
Expand using the distributive property and simplify.

$30 \leq 6x + 12 - 2x - 2 \leq 46$

$30 \leq 4x + 10 \leq 46$

$30 - 10 \leq 4x + 10 - 10 \leq 46 - 10$

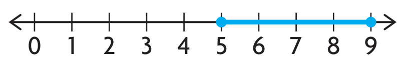
Subtract 10 from all three parts of the inequality.

$20 \leq 4x \leq 36$

$$\frac{20}{4} \leq \frac{4x}{4} \leq \frac{36}{4}$$

Divide all parts of the inequality by 4.

$$5 \leq x \leq 9$$

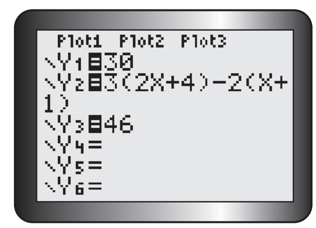


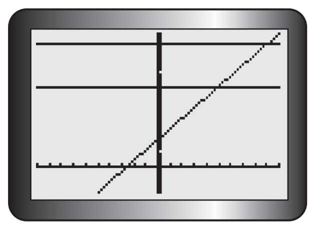
A number line helps to visualize the solution. Solid dots are placed on 5 and 9 since these numbers are included in the solution set.

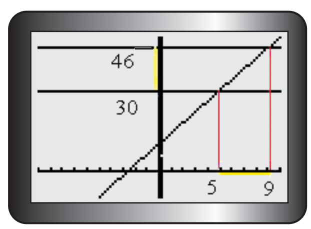
The solution using interval notation is

$x \in [5, 9]$

To verify the solution, graph the functions that correspond to all three parts of the inequality.







The x-values that satisfy the inequality are the x-coordinates of points on the diagonal line defined by $y = 3(2x + 4) - 2(x + 1)$ whose y-values are bounded by 30 and 46.

In Summary

Key Idea

- You can solve a linear inequality using inverse operations in much the same way you solve linear equations.

Need to Know

- If you multiply or divide an inequality by a negative number, you must reverse the inequality sign.
- Most linear equations have only one solution, whereas linear inequalities have many solutions.
- A number line can help you visualize the solution set to an inequality. A solid dot is used to indicate that a number is included in the solution set, whereas an open dot indicates that a number is excluded.

CHECK Your Understanding

1. Solve the following inequalities graphically. Express your answer using set notation.

a) $3x - 1 \leq 11$

b) $-x + 5 > -2$

c) $x - 2 > 3x + 8$

d) $3(2x + 4) \geq 2x$

e) $-2(1 - 2x) < 5x + 8$

f) $\dfrac{6x + 8}{5} \leq 2x - 4$

2. Solve the following inequalities algebraically. Express your answer using interval notation.

a) $2x - 5 \leq 4x + 1$

b) $2(x + 3) < -(x - 4)$

c) $\dfrac{2x + 3}{3} \leq x - 5$

d) $2x + 1 \leq 5x - 2$

e) $-x + 1 > x + 1$

f) $\dfrac{x + 4}{2} \geq \dfrac{x - 2}{4}$

3. Solve the double inequality $3 \leq 2x + 5 < 17$ algebraically and illustrate your solution on a number line.

4. For each of the following inequalities, determine whether $x = 2$ is contained in the solution set.

a) $x > -1$

b) $5x - 4 > 3x + 2$

c) $4(3x - 5) \geq 6x$

d) $5x + 3 \leq -3x + 1$

e) $x - 2 \leq 3x + 4 \leq x + 14$

f) $33 < -10x + 3 < 54$

PRACTISING

5. Solve the following algebraically. Verify your results graphically.

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a) $2x - 1 \leq 13$

b) $-2x - 1 > -1$

c) $2x - 8 > 4x + 12$

d) $5(x - 3) \geq 2x$

e) $-4(5 - 3x) < 2(3x + 8)$

f) $\dfrac{x - 2}{3} \leq 2x - 3$

6. For the following inequalities, determine if 0 is a number in the solution set.

a) $3x \leq 4x + 1$

b) $-6x < x + 4 < 12$

c) $-x + 1 > x + 12$

d) $3x \leq x + 1 \leq x - 1$

e) $x(2x - 1) \leq x + 7$

f) $x + 6 < (x + 2)(5x + 3)$

7. Solve the following inequalities algebraically.

a) $-5 < 2x + 7 < 11$

b) $11 < 3x - 1 < 23$

c) $-1 \leq -x + 9 \leq 13$

d) $0 \leq -2(x + 4) \leq 6$

e) $59 < 7x + 10 < 73$

f) $18 \leq -12(x - 1) \leq 48$

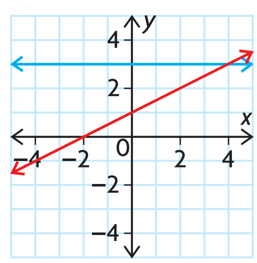
8. a) Create a linear inequality, with both constant and linear terms on each side, for which the solution is $x > 4$.
b) Create a linear inequality, with both constant and linear terms on each side, for which the solution is $x \leq \frac{3}{2}$.

9. The following number line shows the solution to a double inequality.

a) Write the solution using set notation.
b) Create a double inequality for which this is the solution set.

10. Which of the following inequalities has a solution. Explain.
$x - 3 < 3 - x < x - 5$ or $x - 3 > 3 - x > x - 5$

11. Consider the following graph.



a) Write an inequality that is modelled by the graph.
b) Find the solution by examining the graph.
c) Confirm the solution by solving your inequality algebraically.

12. **A** The relationship between Celsius and Fahrenheit is represented by $C = \frac{5}{9}(F - 32)$. In order to be comfortable, but also economical, the temperature in your house should be between 18 °C and 22 °C.
a) Write this statement as a double linear inequality.
b) Solve the inequality to determine the temperature range in degrees Fahrenheit.

13. Some volunteers are making long distance phone calls to raise money for a charity. The calls are billed at the rate of \$0.50 for the first 3 min and \$0.10/min for each additional minute or part thereof. If each call cannot cost more that \$2.00, how long can each volunteer talk to a prospective donor?

14. **a)** Find the equation that allows for the conversion of Celsius to Fahrenheit by solving the relation given in question 12 for F.

b) For what values of C is the Fahrenheit temperature greater than the equivalent Celsius temperature?

15. The inequality $|2x - 1| < 7$ can be expressed as a double inequality.

T **a)** Depict the inequality graphically.

b) Use your graph to solve the inequality.

16. Will the solution to a double inequality always have an upper and
C lower limit? Explain.

Extending

17. Some inequalities are very difficult to solve algebraically. Other methods, however, can be very helpful in solving such problems. Consider the inequality $2^x - 3 < x + 1$.

a) Explain why solving the inequality might be very difficult to do algebraically.

b) Describe an alternative method that could work, and use it to solve the inequality.

18. Some operations result in switching the direction of the inequality when done to both sides, but others result in maintaining the direction. For instance, if you add a constant to both sides, the direction is maintained, whereas multiplying both sides by a negative constant causes the sign to switch. For each of the following, determine if the inequality direction should be maintained, should switch, or if it sometimes switches and sometimes is maintained.

a) cubing both sides

b) squaring both sides

c) making each side the exponent with 2 as the base, i.e., $3 < 5$, so $2^3 < 2^5$

d) making each side the exponent with 0.5 as the base

e) taking the reciprocal of both sides

f) rounding both sides up to the nearest integer

g) taking the square root of both sides

19. Solve each of the following, $x \in \mathbf{R}$. Express your answers using both set and interval notation and graph the solution set on a number line.

a) $x^2 < 4$

b) $4x^2 + 5 \geq 41$

c) $|2x + 2| < 8$

d) $-3x^3 \geq 81$