4.3 Solving Polynomial Inequalities

GOAL

Solve polynomial inequalities.

YOU WILL NEED

- graphing calculator or graphing software

LEARN ABOUT the Math

The elevation of a hiking trail is modelled by the function $h(x) = 2x^3 + 3x^2 - 17x + 12$, where h is the height measured in metres above sea level and x is the horizontal position from a ranger station measured in kilometres. If x is negative, the position is to the west of the station, and if x is positive, the position is to the east. Since the trail extends 4.2 km to the west of the ranger station and 4 km to the east, the model is accurate where $x \in [-4.2, 4]$.

? How can you determine which sections of the trail are above sea level?

EXAMPLE 1 Selecting a strategy to solve the problem

At what distances from the ranger station is the trail above sea level?

Solution A: Using an algebraic strategy and a number line

$2x^3 + 3x^2 - 17x + 12 > 0$ ← The trail is above sea level when the height is positive, i.e., $h(x) > 0$. Write the mathematical model using a **polynomial inequality**.

polynomial inequality

an inequality that contains a polynomial expression (e.g., $5x^3 + 3x^2 - 6x \le 2$)

$$h(1) = 2(1)^3 + 3(1)^2 - 17(1) + 12$$
$$= 0 \text{ so } (x - 1) \text{ is a factor of } h(x)$$

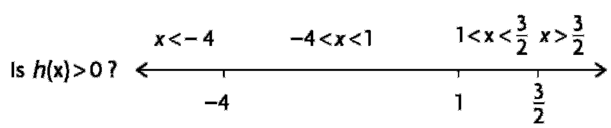
Factor the corresponding function $y = h(x)$ to locate the x-intercepts. Use the factor theorem to determine the first factor.

$$\begin{array}{r|rrrr} 1 & 2 & 3 & -17 & 12 \\ & \downarrow & 2 & 5 & -12 \\ \hline & 2 & 5 & -12 & 0 \end{array}$$

$$h(x) = (x - 1)(2x^2 + 5x - 12)$$
$$0 = (x - 1)(2x - 3)(x + 4)$$
$$x = 1,\ x = \frac{3}{2}, \text{ or } x = -4$$

Set $h(x) = 0$. Set each factor equal to 0 and solve.

The x-intercepts are at -4, 1, and $\frac{3}{2}$. These numbers divide the domain of real numbers into four intervals: $x < -4, -4 < x < 1, 1 < x < \frac{3}{2}, x > \frac{3}{2}$



Draw a number line and test points in each interval to see whether the function has a positive or negative value.

Interval	$x < -4$	$-4 < x < 1$	$1 < x < \frac{3}{2}$	$x > \frac{3}{2}$
Value of $h(x)$	$h(-5) = -78$	$h(0) = 12$	$h(1.2) \doteq -0.6$	$h(2) = 6$
Is $h(x) > 0$?	no	yes	no	yes

$h(x) > 0$ when $-4 < x < 1$ and $x > \frac{3}{2}$.

Identify the intervals where $h(x)$ is positive.

The hiking trail is above sea level from 4 km west of the ranger station to 1 km east, and for distances more than 1.5 km east.

Write a concluding statement.

Solution B: Using a graphing strategy

$$2x^3 + 3x^2 - 17x + 12 > 0$$

The trail is above sea level when the height is positive, i.e., $h(x) > 0$.

$$h(1) = 2(1)^3 + 3(1)^2 - 17(1) + 12$$
$$= 0 \text{ so } (x - 1) \text{ is a factor of } h(x)$$

$$\begin{array}{r|rrrr} 1 & 2 & 3 & -17 & 12 \\ & \downarrow & 2 & 5 & -12 \\ \hline & 2 & 5 & -12 & 0 \end{array}$$

Factor the corresponding function $y = h(x)$ to locate the x-intercepts. Use the factor theorem to determine the first factor.

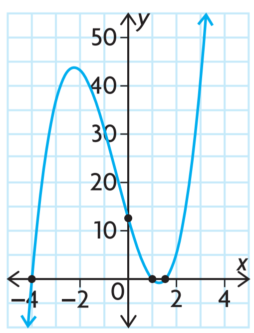
$$h(x) = (x - 1)(2x^2 + 5x - 12)$$
$$0 = (x - 1)(2x - 3)(x + 4)$$
$$x = 1, x = \frac{3}{2}, \text{ or } x = -4$$

Set $h(x) = 0$.
Set each factor equal to 0 and solve.

The x-intercepts are at -4, 1, and $\frac{3}{2}$.

$h(0) = 12$

The y-intercept occurs when $x = 0$.



Analyze the function and draw a sketch of $h(x)$. Plot the x- and y-intercepts. Because the leading coefficient of the function is positive and the degree of the function is odd, the graph has opposite end behaviours. The graph must start in the third quadrant and proceed to the first quadrant. Estimate the location of the turning points.

The graph lies above the x-axis on the intervals $-4 < x < 1$ and $x > \frac{3}{2}$.

Determine the intervals where $h(x) > 0$.

The hiking trail is above sea level from 4 km west of the ranger station to 1 km east, and for distances beyond 1.5 km to the east of the ranger station.

Write a concluding statement that answers the question.

Reflecting

A. When solving a polynomial inequality, which steps are the same as those used when solving a polynomial equation?

B. What additional steps must be taken when solving a polynomial inequality?

C. The zeros of $y = h(x)$ were used to identify the intervals where $h(x)$ was positive and negative but were not included in the solution set of $h(x) > 0$. Explain why.

D. How could you verify the solution set to the polynomial inequality using graphing technology?

APPLY the Math

EXAMPLE 2 Selecting tools and strategies to solve a factorable polynomial inequality

Solve the inequality $x^3 - 2x^2 + 5x + 20 \geq 2x^2 + 14x - 16$.

Solution A: Using algebra and a factor table

$x^3 - 2x^2 + 5x + 20 \geq 2x^2 + 14x - 16$

$x^3 - 4x^2 - 9x + 36 \geq 0$

Use inverse operations to make the right side of the inequality equal to zero.

$x^2(x - 4) - 9(x - 4) \geq 0$

Factor the polynomial on the left by grouping.

$(x - 4)(x^2 - 9) \geq 0$

$(x - 4)(x - 3)(x + 3) \geq 0$

$(x - 4)(x - 3)(x + 3) = 0$

Determine the roots of the corresponding polynomial equation.

The roots are -3, 3, and 4. These numbers divide the real numbers into four intervals:

$x < -3$, $-3 < x < 3$, $3 < x < 4$, $x > 4$

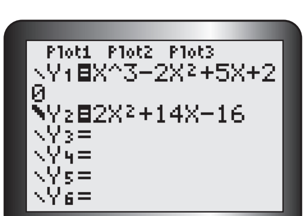
	$x < -3$	$-3 < x < 3$	$3 < x < 4$	$x > 4$
$(x - 4)$	$-$	$-$	$-$	$+$
$(x - 3)$	$-$	$-$	$+$	$+$
$(x + 3)$	$-$	$+$	$+$	$+$
their product	$(-)(-)(-) = -$	$(-)(-)(+) = +$	$(-)(+)(+) = -$	$(+)(+)(+) = +$

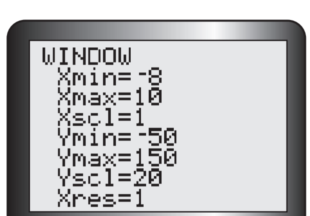
Create a table to consider the sign of each factor in each of the intervals and examine the sign of their product. In this case, the intervals that correspond to a positive product are the solutions to the polynomial inequality.

$x^3 - 2x^2 + 5x + 20 \geq 2x^2 + 14x - 16$ when $-3 \leq x \leq 3$ or $x \geq 4$.

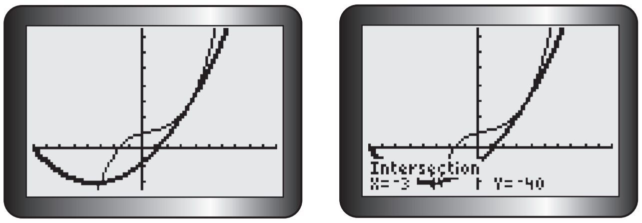
Write a concluding statement.

Solution B: Using graphing technology

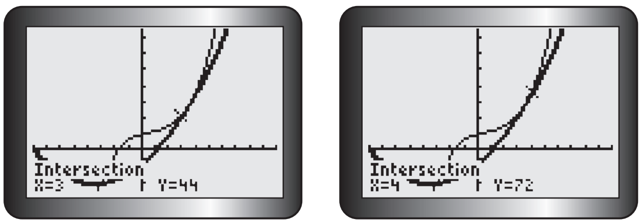




Graph each side of the inequality as a separate function. Bold the graph of the second function (the quadratic) so you can distinguish one from the other. Experiment with different window settings to make the intersecting parts of the graph visible.



The two graphs intersect somewhere to the left of the y-axis and intersect twice to the right of the y-axis. Use the intersect operation to determine all points of intersection.



You can see on the graph that the cubic function lies above the quadratic function in the interval $-3 \leq x \leq 3$ or $x \geq 4$.

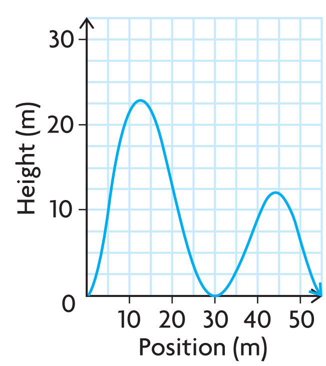
The two functions intersect at $(-3, -40)$, $(3, 44)$, and $(4, 72)$.

$x^3 - 10x^2 + 15x + 11 \geq -x^2 - 8x + 26$
when $x \in [-3, 3]$ or $x \in [4, \infty)$.

Write a concluding statement.

EXAMPLE 3 Selecting a strategy to solve a polynomial inequality that is unfactorable

The height of one section of the roller coaster can be described by the polynomial function $h(x) = \frac{1}{4\,000\,000}x^2(x - 30)^2(x - 55)^2$, where h is the height, measured in metres, and x is the position from the start, measured in metres along the ground.



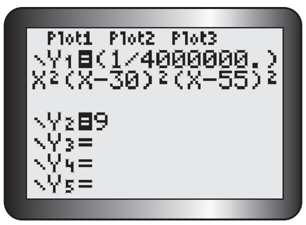
When will the roller coaster car be more than 9 m above the ground?

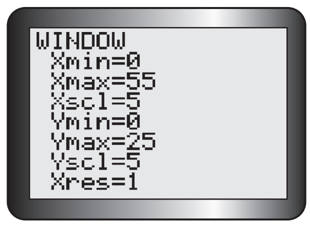
Solution

Solve

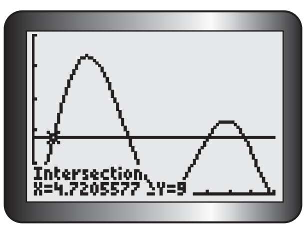
$$\frac{1}{4\ 000\ 000}x^2(x-30)^2(x-55)^2 > 9$$

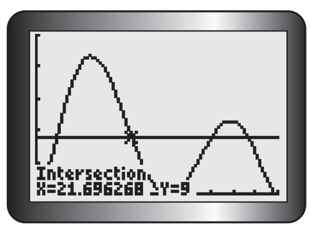
In this case, the solution set corresponds to all values of x where $h(x) > 9$. Using an algebraic approach involving factoring would be tedious, so use a graphing strategy.

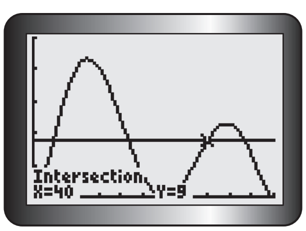


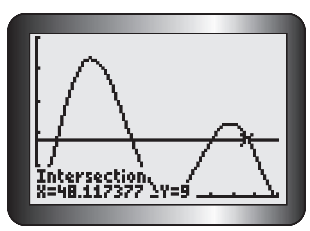


Graph the function $h(x) = \frac{1}{4\ 000\ 000}x^2(x-30)^2(x-55)^2$ and the line $y = 9$ on the graphing calculator and locate intervals where the roller coaster is higher than 9 m. On the graph, this will correspond to when Y1 > Y2.









Determine the four points of intersection of the height function and the horizontal line.

The four points where the height function and the horizontal line intersect are approximately $(4.7, 9)$, $(21.7, 9)$, $(40, 9)$, and $(48.1, 9)$.

Y1 > Y2 when $4.7 < x < 21.7$ or $40 < x < 48.1$.

The roller coaster will be more than 9 m above the ground when it is between 4.7 m and 21.7 m from the starting point and between 40 m and 48.1 m from the starting point, as measured along the ground.

In Summary

Key Idea

- To solve a polynomial inequality algebraically, you must first determine the roots of the corresponding polynomial equation. Then you must consider the sign of the polynomial in each of the intervals created by these roots. The solution set is determined by the interval(s) that satisfy the given inequality.

Need to Know

- Some polynomial inequalities can be solved algebraically by
  - using inverse operations to move all terms to one side of the inequality
  - factoring the polynomial to determine the zeros of the corresponding polynomial equation
  - using a number line, a graph, or a factor table to determine the intervals on which the polynomial is positive or negative
- All polynomial inequalities can be solved using graphing technology by
  - graphing each side of the inequality as a separate function
  - determining the intersection point(s) of the functions
  - examining the graph to determine the intervals where one function is above or below the other, as required

    or
  - creating an equivalent inequality with zero on one side
  - identifying the intervals created by the zeros of the graph of the new function
  - finding where the graph lies above the x-axis (where $f(x) > 0$) or below (where $f(x) < 0$), as required

CHECK Your Understanding

1. Solve each of the following using a number line strategy. Express your answers using set notation.

a) $(x + 2)(x - 3)(x + 1) \geq 0$

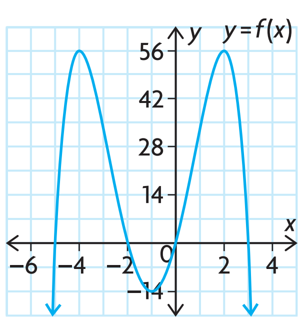
b) $-2(x - 2)(x - 4)(x + 3) < 0$

c) $(x - 3)(5x + 2)(4x - 3) < 0$

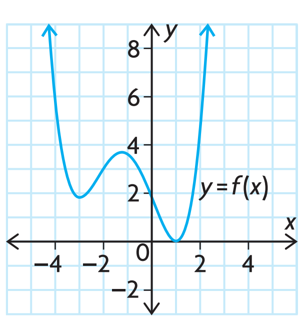
d) $(x - 5)(4x + 1)(2x - 5) \geq 0$

2. For each graph shown, determine where $f(x) \leq 0$. Express your answers using interval notation.

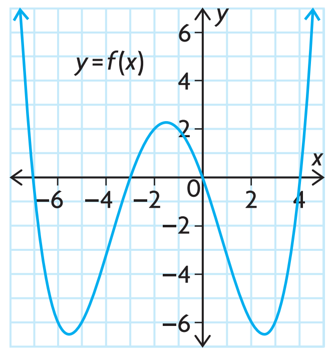
a)



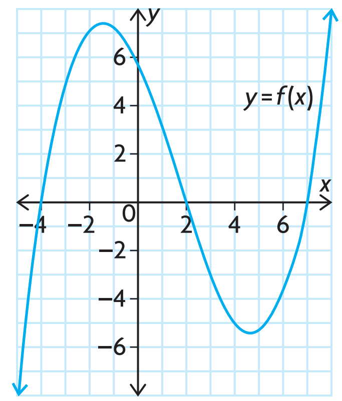
b)



c)



d)



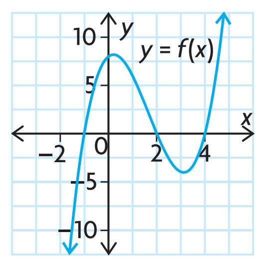
3. If $f(x) = 2x^3 - x^2 + 3x + 10$ and $g(x) = x^3 + 3x^2 + 2x + 4$, determine when $f(x) > g(x)$ using a factor table strategy.

4. Solve the inequality $x^3 - 7x^2 + 4x + 12 > x^2 - 4x - 9$ using a graphing calculator.

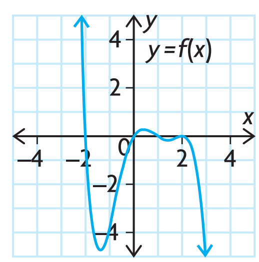
PRACTISING

5. K For each of the following polynomial functions, state the intervals where $f(x) > 0$.

a)



c)



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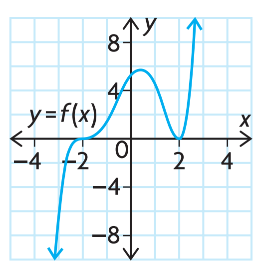
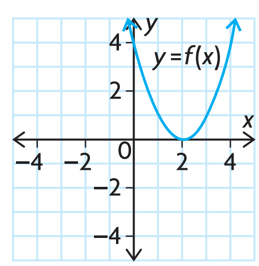


d)



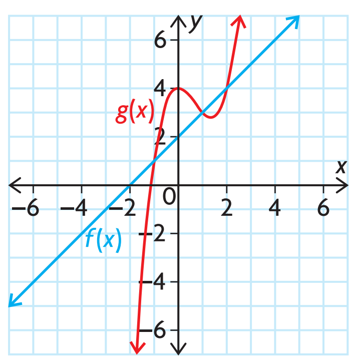
6. Solve the following inequalities.

a) $(x - 1)(x + 1) > 0$
b) $(x + 3)(x - 4) < 0$
c) $(2x + 1)(x - 5) \geq 0$
d) $-3x(x + 7)(x - 2) < 0$
e) $(x - 3)(x + 1) + (x - 3)(x + 2) \geq 0$
f) $2x(x + 4) - 3(x + 4) \leq 0$

7. Solve the following inequalities algebraically. Confirm your answer with a graph.

a) $x^2 - 6x + 9 \geq 16$
b) $x^4 - 8x < 0$
c) $x^3 + 4x^2 + x \leq 6$
d) $x^4 - 5x^2 + 4 > 0$
e) $3x^3 - 3x^2 - 2x \leq 2x^3 - x^2 + x$
f) $x^3 - x^2 - 3x + 3 > -x^3 + 2x + 5$

8. For the following pair of functions, determine when $f(x) < g(x)$.



9. Consider $x^3 + 11x^2 + 18x + 10 > 10$.

a) What is the equation of the corresponding function that could be graphed and used to solve this inequality?
b) Explain how the graph of the corresponding function can be used in this case to solve the inequality.
c) Solve this inequality algebraically.

10. Determine an expression for $f(x)$ in which $f(x)$ is a quartic function, $f(x) > 0$ when $-2 < x < 1$, $f(x) \leq 0$ when $x < -2$ or $x > 1$, $f(x)$ has a double root when $x = 3$, and $f(-1) = 96$.

11. The viscosity, v, of oil used in cars is related to its temperature, t, by the formula $v = -t^3 - 6t^2 + 12t + 50$, where each unit of t is equivalent to 50 °C.

a) Graph the function on your graphing calculator.
b) Determine the temperature range for which $v > 0$ to two decimal places.
c) Determine the temperature ranges for which $15 < v < 20$ to two decimal places.

12. A rock is tossed from a platform and follows a parabolic path through the air. The height of the rock in metres is given by $h(t) = -5t^2 + 12t + 14$, where t is measured in seconds.

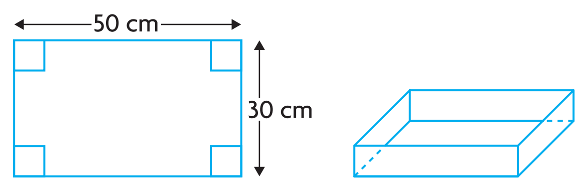
a) How high is the rock off the ground when it is thrown?

b) How long is the rock in the air?

c) For what times is the height of the rock greater than 17 m?

d) How long is the rock above a height of 17 m?

13. **A** An open-topped box can be made from a sheet of aluminium measuring 50 cm by 30 cm by cutting congruent squares from the four corners and folding up the sides. Write a polynomial function to represent the volume of such a box. Determine the range of side lengths that are possible for each square that is cut out and removed that result in a volume greater than 4000 cm^3.



14. **T** **a)** Without a calculator, explain why the inequality $2x^{24} + x^4 + 15x^2 + 80 < 0$ has no solution.

b) Without a calculator, explain why $-4x^{12} - 7x^6 + 9x^2 + 20 < 30 + 11x^2$ has a solution of $-\infty < x < \infty$.

15. Explain why the following solution strategy fails, and then solve the inequality correctly.

Solve: $(x + 1)(x - 2) > (x + 1)(-x + 6)$.
Divide both sides by $x + 1$ and get $x - 2 > -x + 6$.
Add x to both sides: $2x - 2 > 6$.
Add 2 to both sides: $2x > 8$.
Divide both sides by 2: $x > 4$.

16. **C** Create a concept web that illustrates all of the different methods you could use to solve a polynomial inequality.

Extending

17. Use what you know about the factoring method to solve the following inequalities.

a) $\dfrac{x^2 + x - 12}{x^2 + 5x + 6} < 0$ **b)** $\dfrac{x^2 - 25}{x^3 + 6x^2 + 5x} > 0$

18. Solve the inequality $(x + 1)(x - 2)(2^x) \geq 0$ algebraically.