4.4 Rates of Change in Polynomial Functions

GOAL

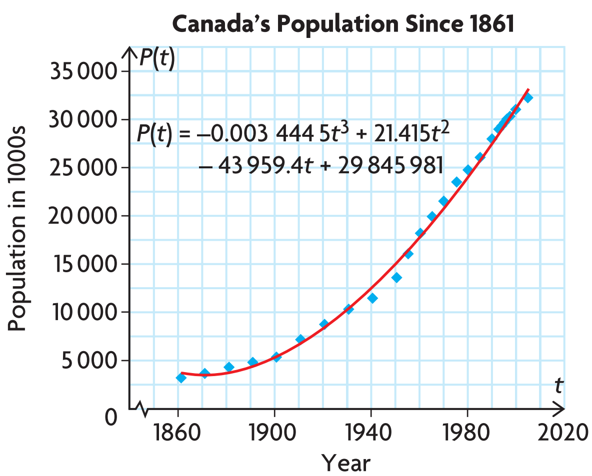
Determine average and instantaneous rates of change in polynomial functions.

YOU WILL NEED

- graphing calculator or graphing software

LEARN ABOUT the Math

Emile is researching Canada's population growth. He obtained the data online and used graphing software to create the following graph and fit a curve to the data.



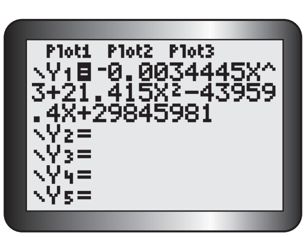
Year	Population (1000s)
1861	3 230
1871	3 689
1881	4 325
1891	4 833
1901	5 371
1911	7 207
1921	8 788
1931	10 377
1941	11 507
1951	13 648
1956	16 081
1961	18 238
1966	20 015
1971	21 568
1976	23 550
1981	24 820
1986	26 101
1991	28 031
1994	29 036
1995	29 354
1996	29 672
1997	30 011
1998	30 301
2001	31 051
2006	32 249

? Was Canada's population growing faster in 1997 or 2005?

EXAMPLE 1 Selecting tools and strategies to determine the instantaneous rate of change

Estimate the instantaneous rates of change in Canada's population in 1997 and 2005, and compare them.

Solution A: Using an algebraic strategy



Enter the equation into the graphing calculator.

Tech *Support*

For help using the graphing calculator to evaluate a function at a given point, see Technical Appendix, T–3.

Average rate of change

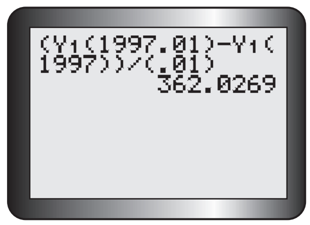
$= \frac{P(a + h) - P(a)}{h}$

$h = 0.01$

$= \frac{P(1997 + 0.01) - P(1997)}{0.01}$

$= \frac{P(1997.01) - P(1997)}{0.01}$

Use the difference quotient and a very small value for h where $a = 1997$ to estimate the instantaneous rate of change in 1997.



Enter the rate of change expression into the graphing calculator to determine its value using the equation entered into Y1.

Average rate of change

$= \frac{P(a + h) - P(a)}{h}$

$h = 0.01$

$= \frac{P(2005 + 0.01) - P(2005)}{0.01}$

$= \frac{P(2005.01) - P(2005)}{0.01}$

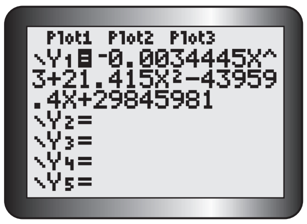
Use the difference quotient and a very small value for h where $a = 2005$ to estimate the instantaneous rate of change in 2005.

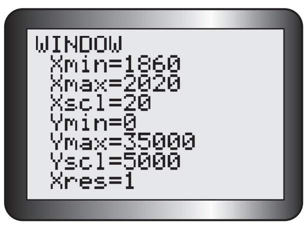
Enter the rate of change expression into the graphing calculator to determine its value using the equation entered into Y1.

The population was increasing by approximately 362 000 people/year in 1997 and 374 000 people/year in 2005. Canada's population was growing faster in 2005.

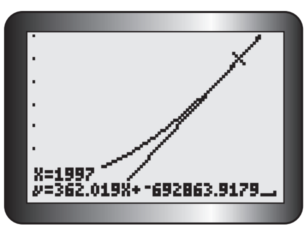
Round off and multiply the rates of change by 1000 since the population is given in thousands.

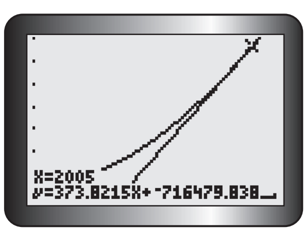
Solution B: Using a graphing strategy





Enter the equation into Y1 on the graphing calculator and adjust the window setting to match the data given.





The instantaneous rate of change at any given point is equal to the slope of the **tangent** line to the curve at that point.
Use the draw tangent operation to draw tangents to the curve at $x = 1997$ and $x = 2005$.
The graphing calculator gives the equation of the tangent in $y = mx + b$ form; thus, the slope of the tangent is the coefficient of x.

Round off and multiply the rate of change by 1000 since the population is given in thousands.

The population was increasing by approximately 362 000 people/year in 1997 and 374 000 people/year in 2005. Canada's population was growing faster in 2005.

Tech Support

For help to draw tangent lines using the graphing calculator's draw operation, see Technical Appendix, T–17.

Reflecting

A. The estimates for the instantaneous rates of change in population for 1997 and 2005 were both positive. Why does this make sense? Explain.

B. Explain how you could determine whether Canada's population was growing faster in 1880 or 1920 by just using the graph that was given.

C. Was Canada's population growing at a constant rate between 1860 and 2006? Explain.

APPLY the Math

EXAMPLE 2 Selecting tools and strategies to determine the slope of a secant

Determine the average rate of change from $x = 2$ to $x = 5$ on the function $f(x) = (x - 3)^3 - 1$.

Solution A: Using an algebraic strategy

Average rate of change

$$= \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Use the average rate of change formula for the interval $2 \le x \le 5$.

$$= \frac{f(5) - f(2)}{5 - 2}$$

$$= \frac{[(5 - 3)^3 - 1] - [(2 - 3)^3 - 1]}{3}$$

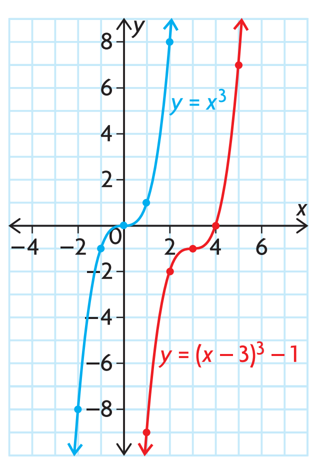
$$= \frac{7 - (-2)}{3}$$

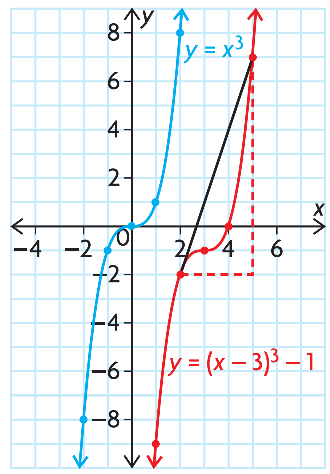
$$= 3$$

Solution B: Using a graphing strategy

$f(x) = (x - 3)^3 - 1$ is a translation right 3 units and down 1 unit of the graph of $y = x^3$.

Use transformations to sketch the graph of the function.



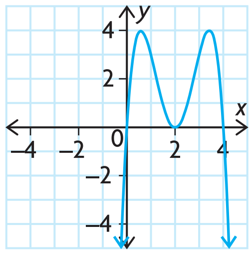


Draw the secant through the points $(2, f(2))$ and $(5, f(5))$, and calculate its slope since the slope of this secant line equals the average rate of change in $f(x)$ on this interval.

$$
\begin{aligned}
m_{\text{secant}} &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{7 - (-2)}{5 - 2} \\
&= \frac{9}{3} \\
&= 3
\end{aligned}
$$

EXAMPLE 3 Selecting tools and strategies to determine the slope of a tangent

The graph of a polynomial function is shown. Estimate the instantaneous rate of change in $f(x)$ at the point $(2, 0)$.



Solution A: Using an algebraic strategy

$$f(x) = ax(x - 2)^2(x - 4)$$

Determine the equation of the polynomial function. The graph has zeros at $x = 0$, 2, and 4. Since the graph is parabolic at $x = 2$, the factor $(x - 2)$ is squared.

$$3 = a(1)(1 - 2)^2(1 - 4)$$
$$3 = a(-3)$$
$$-1 = a$$

Substitute the point $(1, 3)$ into the equation and solve for a.

$$f(x) = -x(x - 2)^2(x - 4)$$

State the equation that represents the function.

$$\text{Slope} = \frac{f(a + h) - f(a)}{h}$$
$$= \frac{f(2 + h) - f(2)}{h}$$

Use the difference quotient to estimate the slope of the tangent line at $(2, 0)$. In this case, $a = 2$ and $f(a) = f(2) = 0$.

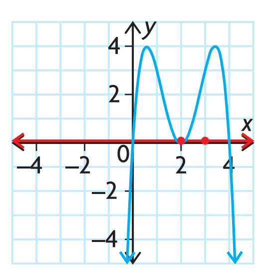
Let $h = 0.001$

$$h = \frac{f(2.001) - f(2)}{0.001}$$
$$h = \frac{[-2.001(2.001 - 2)^2(2.001 - 4)] - 0}{0.001}$$
$$h \doteq \frac{0.000\,004}{0.001}$$
$$h = 0.004$$

The instantaneous rate of change at $(2, 0)$ is approximately 0.

The slope of the tangent line at a turning point on a polynomial function is 0.

Solution B: Using a graphing strategy



Draw the graph on graph paper and sketch the tangent line at the point $A(2, 0)$. Estimate the coordinates of a second point that lies on the tangent line. In this case, use the point $B(3, 0)$.

$$m_{\text{tangent}} = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{0 - 0}{2 - 3}$$
$$= 0$$

Calculate the slope of line AB using the slope formula.

The instantaneous rate of change at $(2, 0)$ is 0.

In Summary

Key Idea

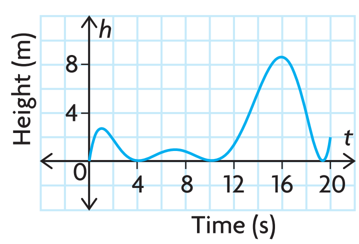
- The methods used previously to calculate average rate of change and estimate instantaneous rate of change can be used with polynomial functions.

Need to Know

- The average rate of change of a polynomial function $y = f(x)$ on the interval from $x_1 \le x \le x_2$ is $\frac{f(x_2) - f(x_1)}{x_2 - x_1}$. Graphically, this is equivalent to the slope of the secant line that passes through the points (x_1, y_1) and (x_2, y_2) on the graph of $y = f(x)$.
- The instantaneous rate of change of a polynomial function $y = f(x)$ at $x = a$ can be approximated by using the difference quotient $\frac{f(a + h) - f(a)}{h}$, where h is a very small value. Graphically, this is equivalent to estimating the slope of the tangent line by calculating the slope of the secant line that passes through the points $(a, f(a))$ and $(a + h, f(a + h))$.
- The instantaneous rate of change of a polynomial function $y = f(x)$ at any of its turning points is 0.

CHECK Your Understanding

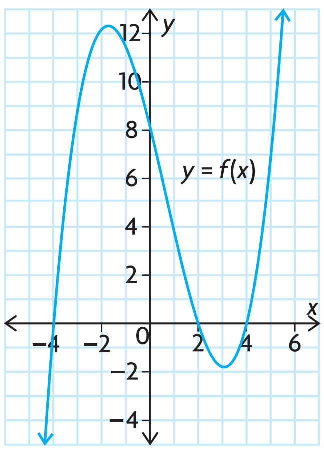
1. Consider the graph showing a bicyclist's elevation relative to his elevation above sea level at the start of the race. The first 20 s of the race are shown.



a) On which intervals will the tangent slope be positive? negative? zero?
b) What do these slopes tell you about the elevation of the bicyclist?

2. Consider the function $f(x) = 3(x - 2)^2 - 2$.
a) Determine the average rate of change in $f(x)$ on each of the following intervals.
i) $2 \le x \le 4$ **ii)** $2 \le x \le 6$ **iii)** $4 \le x \le 6$
b) Estimate the instantaneous rate of change at $x = 4$.
c) Explain why all the rates of change in $f(x)$ calculated in parts a) and b) are positive.
d) State an interval on which the average rate of change in $f(x)$ will be negative.
e) State the coordinates of a point where the instantaneous rate of change in $f(x)$ will be negative.

3. Consider the function $f(x) = x^3 - 4x^2 + 4x$.
   a) Estimate the instantaneous rate of change in $f(x)$ at $x = 2$.
   b) What does your answer to part a) tell you about the graph of the function at $x = 2$?
   c) Sketch a graph of $f(x)$ by first finding the zeros of $f(x)$ to verify your answer to part b).



4. You are given the following graph of $y = f(x)$.
   a) Calculate the average rate of change in $f(x)$ on the interval $4 \leq x \leq 5$.
   b) Estimate the coordinates of the point on the graph of $f(x)$ whose instantaneous rate of change in $f(x)$ is the same as that found in part a).

PRACTISING

5. **K** For each of the following functions, calculate the average rate of change on the interval $x \in [2, 5]$.
   a) $f(x) = 3x + 1$
   b) $t(x) = 3x^2 - 4x + 1$
   c) $g(x) = \frac{1}{x}$
   d) $d(x) = -x^2 + 7$
   e) $h(x) = 2^x$
   f) $v(x) = 9$

6. For each of the functions in question 5, estimate the instantaneous rate of change at $x = 3$.

7. Graph the function $f(x) = x^3 - 2x^2 + x$ by finding its zeros. Use the graph to estimate where the instantaneous rate of change is positive, negative, and zero.

8. A construction worker drops a bolt while working on a high-rise building 320 m above the ground. After t seconds, the bolt's height above the ground is s metres, where $s(t) = 320 - 5t^2$, $0 \leq t \leq 8$.
   a) Find the average velocity for the interval $3 \leq t \leq 8$.
   b) Find the bolt's velocity at $t = 2$.

9. Consider the function $f(x) = 3x^2 - 4x - 1$.
   a) Estimate the slope of the tangent line at $x = 1$.
   b) Find the y-coordinate of the point of tangency.
   c) Use the coordinates of the point of tangency and the slope to find the equation of the tangent line at $x = 1$.

10. **A** The height, h, in metres of a toy rocket above the ground can be modelled by the function $h(t) = -5t^2 + 50t$, where t represents time in seconds.
   a) Use an average speed to approximate the instantaneous speed at $t = 4$.
   b) Use an average speed to approximate the instantaneous speed at $t = 10$.
   c) What is the average speed over the interval from $t = 0$ s to $t = 10$?

11. The distance in kilometres of a boat from its dock can be modelled by the function $d(t) = \left(\frac{1}{200}\right)t^2(t - 8)^2$, where t is in minutes and $t \in [0, 8]$. Sketch a graph that models this situation.

a) Estimate when the instantaneous rate of change in distance to the dock is positive, negative, and zero.

b) What happens to the boat when the instantaneous rate of change in distance is zero? What does it mean when the boat's rate of change in distance is negative?

12. Approximate the instantaneous rate of change at the zeros of the following function: $y = x^4 - 2x^3 - 8x^2 + 18x - 9$.

13. Consider the function $f(x) = x^2 + 3x - 5$.

T **a)** Estimate the instantaneous rate of change at $x = 1$.

b) Simplify the expression $\frac{f(x + h) - f(x)}{h}$.

c) Examine the expression in part b) and discuss what happens as h becomes very close to 0.

d) Use your result from part c) to come up with an expression for the instantaneous rate of change at the point x, and check your result from part a) using the expression.

14. Explain how instantaneous rates of change could be used to locate the
C local maxima and local minima for a polynomial function.

Extending

15. Consider the function $f(x) = e^x$ (e is called Euler's Number where $e \doteq 2.7183$).

a) Estimate the instantaneous rate of change at $x = 5$. Find $f(5)$.

b) Repeat part a) with three more x-values.

c) Generalize your findings.

16. Consider the function $f(x) = x^3 - 4x$.

a) Estimate the slope of the tangent line at $x = 1$.

b) Using the slope and the point of tangency, find the equation of the tangent line.

c) The tangent line intersects the original graph one more time. Where? Graph both the original function and the tangent line to illustrate this.

17. Determine, to two decimal places, where the slope of a tangent line and the slope of the secant line that passes through $A(2, -4)$ and $B(3, 0)$ are equal on the graph of $f(x) = x^3 - 3x^2$.