5.3 Graphs of Rational Functions of the Form $f(x) = \frac{ax + b}{cx + d}$

GOAL

Sketch the graphs of rational functions, given equations of the form $f(x) = \frac{ax + b}{cx + d}$.

YOU WILL NEED

- graph paper
- graphing calculator or graphing software

INVESTIGATE the Math

The radius, in centimetres, of a circular juice blot on a piece of paper towel is modelled by $r(t) = \frac{1 + 2t}{1 + t}$, where t is measured in seconds. According to this model, the maximum size of the blot is determined by the location of the horizontal asymptote.

? How can you find the equation of the horizontal asymptote of a rational function of the form $f(x) = \frac{ax + b}{cx + d}$?

A. Without graphing, determine the domain, intercepts, vertical asymptote, and positive/negative intervals of the simple rational function $f(x) = \frac{x}{x + 1}$.

B. Copy the following tables, and complete them by evaluating $f(x)$ for each value of x. Examine the **end behaviour** of $f(x)$ by observing the trend in $f(x)$ as x grows positively large and negatively large. What value does $f(x)$ seem to approach?

$x \to \infty$	
x	$f(x) = \frac{x}{x + 1}$
10	
100	
1 000	
10 000	
100 000	
1 000 000	

$x \to -\infty$	
x	$f(x) = \frac{x}{x + 1}$
−10	
−100	
−1 000	
−10 000	
−100 000	
−1 000 000	

C. Write an equation for the horizontal asymptote of the function in part B.

D. Repeat parts A, B, and C for the functions $g(x) = \frac{4x}{x + 1}$, $h(x) = \frac{2x}{3x + 1}$, and $m(x) = \frac{3x - 2}{2x - 5}$.

E. Verify your results by graphing all the functions in part D on a graphing calculator. Note similarities and differences among the graphs.

F. Make a list of the equations of the functions and the equations of their horizontal asymptotes. Discuss how the degree of the numerator compares with the degree of the denominator. Explain how the leading coefficients of x in the numerator and the denominator determine the equation of the horizontal asymptote.

G. Determine the equation of the horizontal asymptote of the juice blot function $r(t) = \frac{1 + 2t}{1 + t}$. What does this equation tell you about the eventual size of the juice blot?

Reflecting

H. How do the graphs of rational functions with linear expressions in the numerator and denominator compare with the graphs of reciprocal functions?

I. Explain how you determined the equation of a horizontal asymptote from
   i) end behaviour tables
   ii) the equation of the function

APPLY the Math

EXAMPLE 1 Selecting a strategy to determine how a graph approaches a vertical asymptote

Determine how the graph of $f(x) = \frac{3x - 5}{x + 2}$ approaches its vertical asymptote.

Solution

$f(x) = \frac{3x - 5}{x + 2}$ has a vertical asymptote with the equation $x = -2$. Near this asymptote, the values of the function will grow very large in a positive direction or very large in a negative direction.

$f(x)$ is undefined when $x = -2$.

There is no common factor in the numerator and denominator.

Choose a value of x to the left and very close to -2. This value is less than -2.

$$f(-2.1) = \frac{3(-2.1) - 5}{(-2.1) + 2}$$
$$= 113$$

The graph of a rational function never crosses a vertical asymptote, so choose x-values that are very close to the vertical asymptote, on both sides, to determine the behaviour of the function.

On the left side of the vertical asymptote, the values of the function are positive. As $x \to -2$, $f(x) \to \infty$.

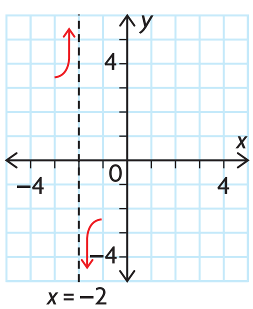
The function increases to large positive values as x approaches -2 from the left.

Choose a value of x to the right and very close to -2. This value is greater than -2.

$$f(-1.9) = \frac{3(-1.9) - 5}{(-1.9) + 2}$$
$$= -107$$

On the right side of the vertical asymptote, the values of the function are negative. As $x \to -2$, $f(x) \to -\infty$.

The function decreases to small negative values as x approaches -2 from the right.



Make a sketch to show how the graph approaches the vertical asymptote.

EXAMPLE 2 Using key characteristics to sketch the graph of a rational function

For each function,

a) $f(x) = \frac{2}{x - 3}$ **b)** $f(x) = \frac{x - 2}{3x + 4}$ **c)** $f(x) = \frac{x - 3}{2x - 6}$

i) determine the domain, intercepts, asymptotes, and positive/negative intervals
ii) use these characteristics to sketch the graph of the function
iii) describe where the function is increasing or decreasing

Solution

a) $f(x) = \dfrac{2}{x - 3}$

i) $D = \{x \in \mathbf{R} | x \neq 3\}$

The function $f(x) = \frac{2}{x - 3}$ is undefined when $x = 3$.

$f(0) = -\frac{2}{3}$, so the y-intercept is $-\frac{2}{3}$.

$f(x) \neq 0$, so there is no x-intercept.

Any rational function equals zero when its numerator equals zero. The numerator is always 2, so $f(x)$ can never equal zero.

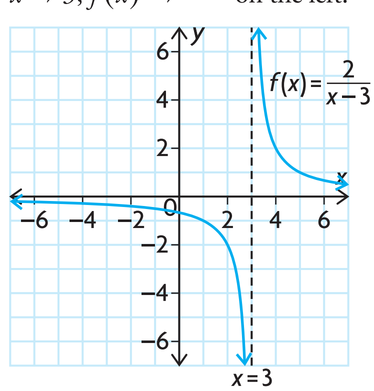
The line $x = 3$ is a vertical asymptote.
The line $y = 0$ is a horizontal asymptote.

Since the numerator and denominator do not contain the common factor $(x - 3)$, $f(x)$ has a vertical asymptote at $x = 3$. Any rational function that is formed by a constant numerator and a linear function denominator has a horizontal asymptote at $y = 0$.

$f(x)$ is negative when $x \in (-\infty, 3)$ and positive when $x \in (3, \infty)$.

The numerator is always positive, so the denominator determines the sign of $f(x)$.
$x - 3 < 0$ when $x < 3$
$x - 3 > 0$ when $x > 3$

ii) Confirm the behaviour of $f(x)$ near the vertical asymptote.
$f(3.1) = 20$, so as $x \to 3, f(x) \to \infty$ on the right.
$f(2.9) = -20$, so as $x \to 3, f(x) \to -\infty$ on the left.



Use all the information in part i) to sketch the graph.

iii) From the graph, the function is decreasing on its entire domain: when $x \in (-\infty, 3)$ and when $x \in (3, \infty)$.

b) $f(x) = \dfrac{x - 2}{3x + 4}$

i) $3x + 4 \neq 0$

$3x \neq -4$

$x \neq -\dfrac{4}{3}$

$D = \left\{x \in \mathbf{R} \mid x \neq -\dfrac{4}{3}\right\}$

$f(x)$ is undefined when the denominator is zero.

$f(0) = \dfrac{0 - 2}{3(0) + 4} = \dfrac{-2}{4}$ or $-\dfrac{1}{2}$

To determine the y-intercept, let $x = 0$.

The y-intercept is $-\dfrac{1}{2}$.

$f(x) = 0$ when $\dfrac{x - 2}{3x + 4} = 0.$

$x - 2 = 0$

$x = 2$

To determine the x-intercept, let $y = 0$. Any rational function equals zero when its numerator equals zero.

The x-intercept is 2.

The line $x = -\frac{4}{3}$ is a vertical asymptote.

This is the value that makes $f(x)$ undefined.

The line $y = \frac{1}{3}$ is a horizontal asymptote.

The ratio of the leading coefficients of the numerator and denominator is $\frac{1}{3}$.

Examine the signs of the numerator and denominator, and their quotient, to determine the positive/negative intervals.

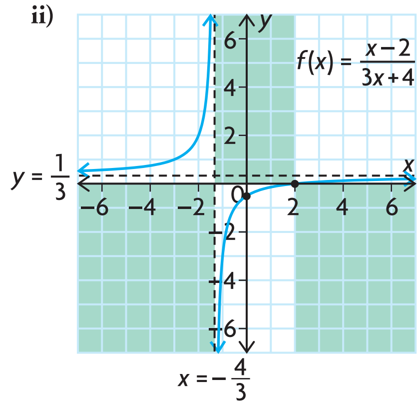
	$x < -\frac{4}{3}$	$-\frac{4}{3} < x < 2$	$x > 2$
$x - 2$	$-$	$-$	$+$
$3x + 4$	$-$	$+$	$+$
$\dfrac{x - 2}{3x - 4}$	$\frac{-}{-} = +$	$\frac{-}{+} = -$	$\frac{+}{+} = +$

The vertical asymptote and the x-intercept divide the set of real numbers into three intervals: $\left(-\infty, -\frac{4}{3}\right)$, $\left(-\frac{4}{3}, 2\right)$, and $(2, \infty)$. Choose numbers in each interval to evaluate the sign of each expression.

$f(x)$ is positive when $x \in \left(-\infty, -\frac{4}{3}\right)$ and when $x \in (2, \infty)$.

$f(x)$ is negative when $x \in \left(-\frac{4}{3}, 2\right)$.

ii)



When sketching the graph, it helps to shade the regions where there is no graph. Use the positive and negative intervals as indicators for these regions. For example, since $f(x)$ is positive on $\left(-\infty, -\frac{4}{3}\right)$, there is no graph under the x-axis on this interval. Draw the asymptotes, and mark the intercepts. Then draw the graph to approach the asymptotes.

iii) From the graph, $f(x)$ is increasing on its entire domain; that is, when $x \in \left(-\infty, -\frac{4}{3}\right)$ and when $x \in \left(-\frac{4}{3}, \infty\right)$.

c) $f(x) = \frac{x - 3}{2x - 6}$

i) $2x - 6 \neq 0$

$2x \neq 6$

$x \neq 3$

$D = \{x \in \mathbf{R} | x \neq 3\}$

$f(x)$ is undefined when the denominator is zero.

$f(0) = \frac{0 - 3}{2(0) - 6} = \frac{-3}{-6}$ or $\frac{1}{2}$

To determine the y-intercept, let $x = 0$.

The y-intercept is $\frac{1}{2}$.

$f(x) \neq 0$, so there is no x-intercept.

To determine the x-intercept, let $y = 0$. Only consider when the numerator is zero; that is, when $x - 3 = 0$. Therefore, the numerator is zero at $x = 3$, but this has already been excluded from the domain.

$f(x) = \frac{x - 3}{2(x - 3)}$

$f(x)$ has a hole, not a vertical asymptote, where $x = 3$.

Factoring reveals a common factor $(x - 3)$ in the numerator and denominator. The graph has a hole at the point where $x = 3$.

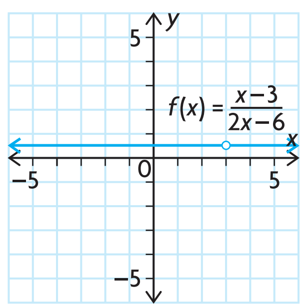
$f(x) = \frac{\cancel{x - 3}^{1}}{2(\cancel{x - 3})_{1}} = \frac{1}{2}$

The value of the function is always $\frac{1}{2}$ for all values of x, except when $x = 3$.

$f(x)$ is positive at all points in its domain.

$f(x)$ has no vertical asymptote or x-intercept. There is only one interval to consider: $(-\infty, \infty)$. For any value of x, $f(x) = \frac{1}{2}$.

ii) The graph is a horizontal line with the equation $y = \frac{1}{2}$. There is a hole at $x = 3$.



Use the information in part i) to sketch the graph.

iii) The function is neither increasing nor decreasing. It is constant on its entire domain.

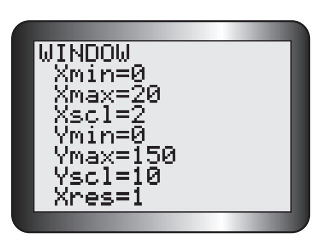
EXAMPLE 3 Solving a problem by graphing a rational function

The function $P(t) = \frac{30(7t + 9)}{3t + 2}$ models the population, in thousands, of a town t years since 1990. Describe the population of the town over the next 20 years.

Solution

$$P(t) = \frac{30(7t + 9)}{3t + 2}$$

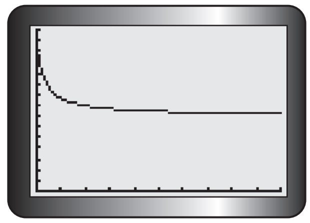
Use the equation to help you decide on the window settings. For the given context, $t \geq 0$ and $P(t) > 0$.



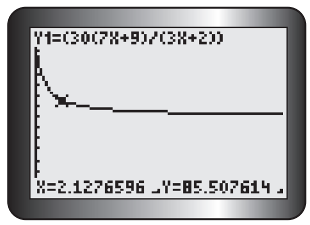
Determine the initial population in 1990, when $t = 0$.

$$P(0) = \frac{30(7(0) + 9)}{3(0) + 2} = 135$$

Graph $P(t)$ to show the population for the 20 years after 1990.



The value that makes $3t + 2 = 0$ lies outside the domain of $P(t)$. There is no vertical asymptote in the domain.

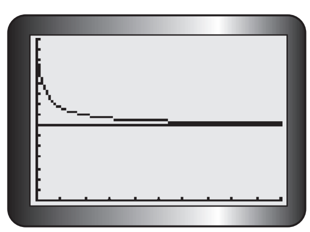


TRACE along the curve to get an idea of how the population changed.

In the first two years, the population dropped by about 50 000 people. Then it began to level off and approach a steady value.

There is a horizontal asymptote at $P = \frac{30(7)}{3} = 70$.

Use the function equation to determine the equation of the horizontal asymptote. For large values of t, $P(t) \doteq \frac{30(7t)}{3t}$. Therefore, the leading coefficients in the numerator and denominator define the equation of the horizontal asymptote.



The population of the town has been decreasing since 1990. It was 135 000 in 1990, but dropped by about 50 000 in the next two years. Since then, the population has begun to level off and, according to the model, will approach a steady value of 70 000 people by 2010.

Multiply the values of the function by 1000, since the population is given in thousands.

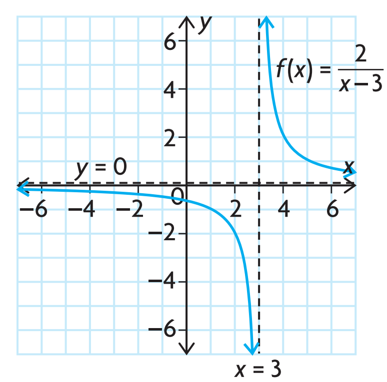
In Summary

Key Ideas

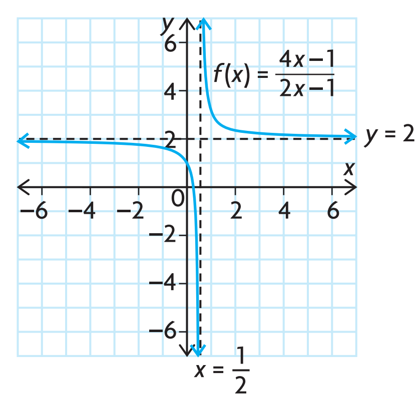
- The graphs of most rational functions of the form $f(x) = \frac{b}{cx + d}$ and $f(x) = \frac{ax + b}{cx + d}$ have both a vertical asymptote and a horizontal asymptote.
- You can determine the equation of the vertical asymptote directly from the equation of the function by finding the zero of the denominator.
- You can determine the equation of the horizontal asymptote directly from the equation of the function by examining the ratio of the leading coefficients in the numerator and the denominator. This gives you the end behaviours of the function.
- To sketch the graph of a rational function, you can use the domain, intercepts, equations of asymptotes, and positive/negative intervals.

Need to Know

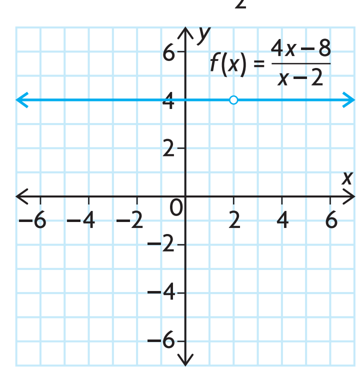
- Rational functions of the form $f(x) = \frac{b}{cx + d}$ have a vertical asymptote defined by $x = -\frac{d}{c}$ and a horizontal asymptote defined by $y = 0$. For example, see the graph of $f(x) = \frac{2}{x - 3}$.



- Most rational functions of the form $f(x) = \frac{ax + b}{cx + d}$ have a vertical asymptote defined by $x = -\frac{d}{c}$ and a horizontal asymptote defined by $y = \frac{a}{c}$. For example, see the graph of $f(x) = \frac{4x - 1}{2x - 1}$.



The exception occurs when the numerator and the denominator both contain a common linear factor. This results in a graph of a horizontal line that has a hole where the zero of the common factor occurs. As a result, the graph has no asymptotes. For example, see the graph of $f(x) = \frac{4x - 8}{x - 2} = \frac{4(x - 2)}{(x - 2)}$.



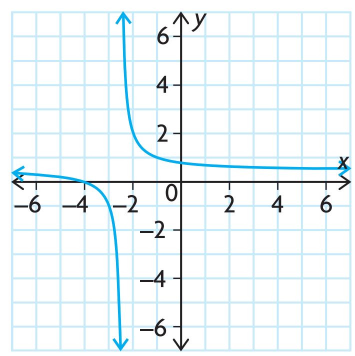
CHECK *Your Understanding*

1. Match each function with its graph.

a) $h(x) = \dfrac{x + 4}{2x + 5}$ **c)** $f(x) = \dfrac{3}{x - 1}$

b) $m(x) = \dfrac{2x - 4}{x - 2}$ **d)** $g(x) = \dfrac{2x - 3}{x + 2}$

A



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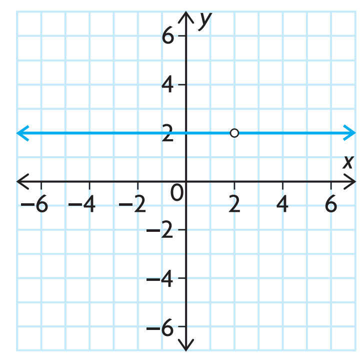

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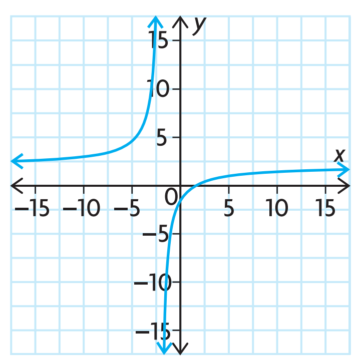

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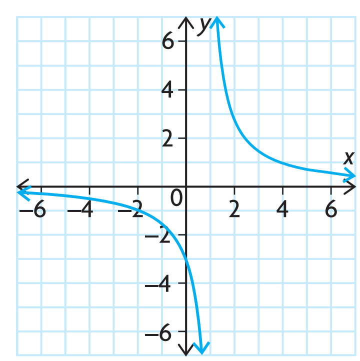

2. Consider the function $f(x) = \frac{3}{x - 2}$.

a) State the equation of the vertical asymptote.

b) Use a table of values to determine the behaviour(s) of the function near its vertical asymptote.

c) State the equation of the horizontal asymptote.

d) Use a table of values to determine the end behaviours of the function near its horizontal asymptote.

e) Determine the domain and range.

f) Determine the positive and negative intervals.

g) Sketch the graph.

3. Repeat question 2 for the rational function $f(x) = \frac{4x - 3}{x + 1}$.

PRACTISING

4. State the equation of the vertical asymptote of each function. Then choose a strategy to determine how the graph of the function approaches its vertical asymptote.

a) $y = \frac{2}{x + 3}$ **c)** $y = \frac{2x + 1}{2x - 1}$

b) $y = \frac{x - 1}{x - 5}$ **d)** $y = \frac{3x + 9}{4x + 1}$

5. K For each function, determine the domain, intercepts, asymptotes, and positive/negative intervals. Use these characteristics to sketch the graph of the function. Then describe where the function is increasing or decreasing.

a) $f(x) = \frac{3}{x + 5}$ **c)** $f(x) = \frac{x + 5}{4x - 1}$

b) $f(x) = \frac{10}{2x - 5}$ **d)** $f(x) = \frac{x + 2}{5(x + 2)}$

6. Read each set of conditions. State the equation of a rational function of the form $f(x) = \frac{ax + b}{cx + d}$ that meets these conditions, and sketch the graph.

a) vertical asymptote at $x = -2$, horizontal asymptote at $y = 0$; negative when $x \in (-\infty, -2)$, positive when $x \in (-2, \infty)$; always decreasing

b) vertical asymptote at $x = -2$, horizontal asymptote at $y = 1$; x-intercept $= 0$, y-intercept $= 0$; positive when $x \in (-\infty, -2)$ or $(0, \infty)$, negative when $x \in (-2, 0)$

c) hole at $x = 3$; no vertical asymptotes; y-intercept $= (0, 0.5)$

d) vertical asymptotes at $x = -2$ and $x = 6$, horizontal asymptote at $y = 0$; positive when $x \in (-\infty, -2)$ or $(6, \infty)$, negative when $x \in (-2, 6)$; increasing when $x \in (-\infty, 2)$, decreasing when $x \in (2, \infty)$

7. T **a)** Use a graphing calculator to investigate the similarities and differences in the graphs of rational functions of the form $f(x) = \frac{8x}{nx + 1}$, for $n = 1, 2, 4$, and 8.

b) Use your answer for part a) to make a conjecture about how the function changes as the values of n approach infinity.

c) If n is negative, how does the function change as the value of n approaches negative infinity? Choose your own values, and use them as examples to support your conclusions.

8. Without using a graphing calculator, compare the graphs of the rational functions $f(x) = \frac{3x + 4}{x - 1}$ and $g(x) = \frac{x - 1}{2x + 3}$.

9. The function $I(t) = \frac{15t + 25}{t}$ gives the value of an investment, in thousands of dollars, over t years.
   a) What is the value of the investment after 2 years?
   b) What is the value of the investment after 1 year?
   c) What is the value of the investment after 6 months?
   d) There is an asymptote on the graph of the function at $t = 0$. Does this make sense? Explain why or why not.
   e) Choose a very small value of t (a value near zero). Calculate the value of the investment at this time. Do you think that the function is accurate at this time? Why or why not?
   f) As time passes, what will the value of the investment approach?

10. **A** An amount of chlorine is added to a swimming pool that contains pure water. The concentration of chlorine, c, in the pool at t hours is given by $c(t) = \frac{2t}{2 + t}$, where c is measured in milligrams per litre. What happens to the concentration of chlorine in the pool during the 24 h period after the chlorine is added?

11. **C** Describe the key characteristics of the graphs of rational functions of the form $f(x) = \frac{ax + b}{cx + d}$. Explain how you can determine these characteristics using the equations of the functions. In what ways are the graphs of all the functions in this family alike? In what ways are they different? Use examples in your comparison.

Extending

12. Not all asymptotes are horizontal or vertical. Find a rational function that has an asymptote that is neither horizontal nor vertical, but slanted or oblique.

13. Use long division to rewrite $f(x) = \frac{2x^3 - 7x^2 + 8x - 5}{x - 1}$ in the form $f(x) = ax^2 + bx + c + \frac{k}{x - 1}$. What does this tell you about the end behaviour of the function? Graph the function. Include all asymptotes in your graph. Write the equations of the asymptotes.

14. Let $f(x) = \frac{3x - 1}{x^2 - 2x - 3}$, $g(x) = \frac{x^3 + 8}{x^2 + 9}$, $h(x) = \frac{x^3 - 3x}{x + 1}$, and $m(x) = \frac{x^2 + x - 12}{x^2 - 4}$.
   a) Which of these rational functions has a horizontal asymptote?
   b) Which has an oblique asymptote?
   c) Which has no vertical asymptote?
   d) Graph $y = m(x)$, showing the asymptotes and intercepts.