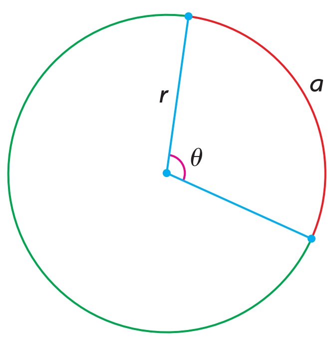
6.1 Radian Measure

GOAL

Use radian measurement to represent the size of an angle.

LEARN ABOUT *the Math*



Angles are commonly measured in degrees. In mathematics and physics, however, there are many applications in which expressing the size of an angle as a pure number, without units, is more convenient than using degrees. In these applications, the size of an angle is expressed in terms of the length of an arc, a, that subtends the angle, θ, at the centre of a circle with radius r. In this situation, a is proportional to r and also to θ, where $\theta = \frac{a}{r}$. The unit of measure is the radian.

radian

the size of an angle that is subtended at the centre of a circle by an arc with a length equal to the radius of the circle; both the arc length and the radius are measured in units of length (such as centimetres) and, as a result, the angle is a real number without any units

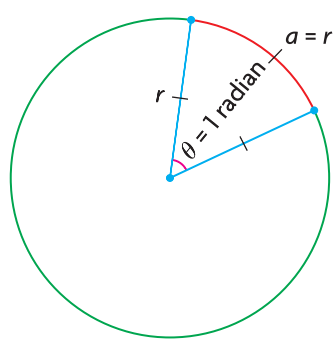
$\theta = \frac{a}{r} = \frac{r}{r} = 1$

❓ How are radians and degrees related to each other?

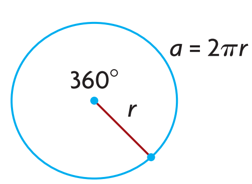
EXAMPLE 1 Connecting radians and degrees

How many degrees is 1 radian?

Solution



1 radian is defined as the angle subtended by an arc length, a, equal to the radius, r. It appears as though 1 radian should be a little less than 60°, since the sector formed resembles an equilateral triangle, with one side that is curved slightly.



Consider the arc length created by an angle of 360°. This arc length is $2\pi r$, the circumference of the circle.

$$\theta = \frac{2\pi \cancel{r}^{1}}{\cancel{r}_{1}} = 2\pi \text{ radians}$$

Using the relationship $\theta = \frac{a}{r}$, the size of the angle can be expressed in radians.

$\theta = 360°$

θ is also 360°, the size of the central angle with an arc length of $2\pi r$.

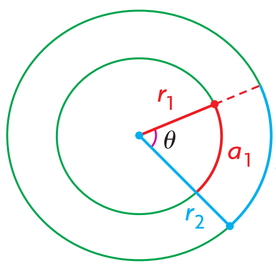
$$2\pi \text{ radians} = 360°$$
$$\pi \text{ radians} = 180°$$

Equating the measures of θ gives an expression relating degrees and radians.

Dividing both sides by 2 gives a simpler relationship between degrees and radians.

$$\frac{\pi \text{ radians}}{\pi} = \frac{180°}{\pi}$$
$$1 \text{ radian} = \frac{180°}{\pi} \doteq 57.3°$$

Dividing both sides by π gives the value of 1 radian in degree measure.



It is important to note that the size of an angle in radians is not affected by the size of the circle. The diagram shows that a_1 and a_2 subtend the same angle θ, so $\theta = \frac{a_1}{r_1} = \frac{a_2}{r_2}$.

The relationship π radians = 180° can be used to convert between degrees and radians.

EXAMPLE 2 Reasoning how to convert degrees to radians

Convert each of the following angles to radians.

a) 20° **b)** 225°

Solution

a) $\pi \text{ radians} = 180°$

Divide both sides by 180° to get an equivalent expression that is equal to 1.

$$\frac{\pi}{180°} = 1$$

Multiplying by 1 creates an equivalent expression, so multiply by $\frac{\pi}{180°}$ to convert degrees to radians.

$$20° = (\overset{1}{\cancel{20°}})\left(\frac{\pi}{\underset{9}{\cancel{180°}}}\right)$$

Simplify by dividing by the common factor of 20. Notice that the units cancel out.

$$= \frac{\pi}{9}$$
$$\doteq 0.35$$

Express the answer as an exact value in terms of π or as an approximate decimal value, as required.

Communication *Tip*

Whenever an angle is expressed without a unit (that is, as a real number), it is understood to be in radians. We often write "radians" after the number, as a reminder that we are discussing an angle.

b) $225° = (225°)\left(\frac{\pi}{180°}\right)$ ← Multiply by $\frac{\pi}{180°}$ to convert degrees to radians.

$= \frac{225°\pi}{180°} \div \frac{45}{45}$ ← Simplify by dividing by the common factor of 45. (Note that the degree symbols cancel.)

$= \frac{5\pi}{4}$

EXAMPLE 3 Reasoning how to convert radians to degrees

Convert each radian measure to degrees.

a) $\frac{5\pi}{6}$ **b)** 1.75 radians

Solution

a) π radians $= 180°$

$\frac{5\pi}{6} = \frac{5(180°)}{6}$ ← Substitute 180° for π.

$= 5(30°)$ ← Evaluate.

$= 150°$

b) π radians $= 180°$

$1 = \frac{180°}{\pi \text{ radians}}$ ← Divide both sides by π radians to get an equivalent expression that is equal to 1.

$1.75 \text{ radians} = 1.75\ \overset{1}{\cancel{\text{radians}}} \times \frac{180°}{\pi\ \underset{1}{\cancel{\text{radians}}}}$ ← Multiplying by 1 creates an equivalent expression, so multiply by $\frac{180°}{\pi \text{ radians}}$ to convert radians to degrees.

$\doteq 100.3°$

Reflecting

A. Consider the formula $\theta = \frac{a}{r}$. Explain why angles can be described as having no unit when they are measured in radians.

B. Explain how to convert any angle measure that is given in degrees to radians.

C. Explain how to convert any angle measure that is given in radians to degrees.

APPLY the Math

EXAMPLE 4 Solving a problem that involves radians

The London Eye Ferris wheel has a diameter of 135 m and completes one revolution in 30 min.

a) Determine the angular velocity, ω, in radians per second.

b) How far has a rider travelled at 10 min into the ride?

Solution

a) $30 \text{ min} = 30 \cancel{\text{min}} \times \dfrac{60 \text{ s}}{1 \cancel{\text{min}}}$

$= 1800 \text{ s}$

Since the question asks for angular velocity in radians per second, convert the time to seconds.

Angular velocity, $\omega = \dfrac{2\pi}{1800}$ radians/s

$= \dfrac{\pi}{900}$ radians/s

$\doteq 0.003\ 49$ radians/s

Each revolution of the Ferris wheel represents an angular motion through an angle of 2π radians. Therefore, the Ferris wheel moves through 2π radians every 30 min.

b) Radius, $r = \dfrac{135}{2}$ m

$= 67.5$ m

The rider moves in a circular motion on the edge of a circle that has a radius of 67.5 m.

Number of revolutions, $n = \dfrac{10 \cancel{\text{min}}}{30 \cancel{\text{min}}}$

$= \dfrac{1}{3}$ revolution

The wheel turns through one revolution every 30 min, so the rider has gone through $\frac{1}{3}$ of a revolution at 10 min.

Distance travelled, $d = \dfrac{1}{3}(2\pi \times 67.5 \text{ m})$

$= 45\pi$ m

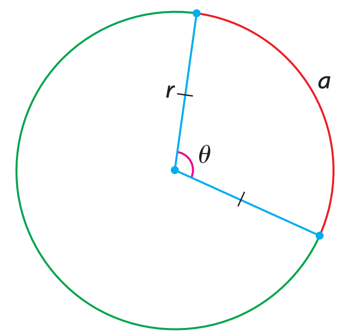
$\doteq 141.4$ m

The rider travels $\frac{1}{3}$ of the circumference in 10 min.

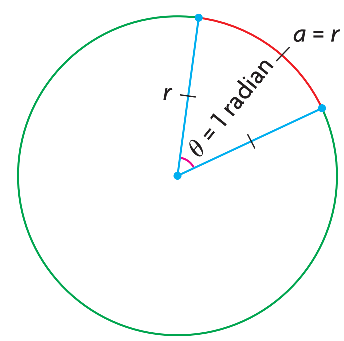
In Summary

Key Ideas

- The radian is an alternative way to represent the size of an angle. The arc length, a, of a circle is proportional to its radius, r, and the central angle that it subtends, θ, by the formula $\theta = \frac{a}{r}$.



- One radian is defined as the angle subtended by an arc that is the same length as the radius. $\theta = \frac{a}{r} = \frac{r}{r} = 1$. 1 radian is about 57.3°.

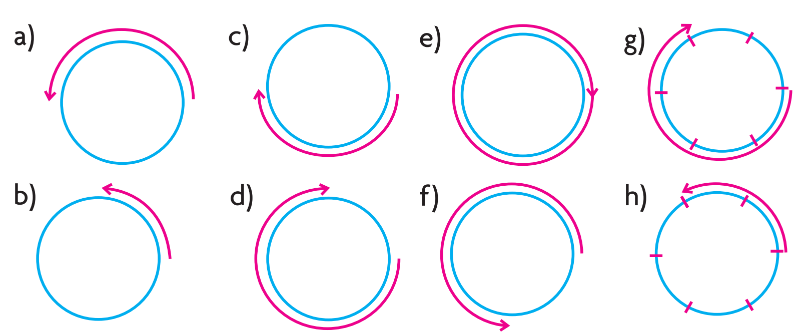


Need to Know

- Using radians enables you to express the size of an angle as a real number without any units, often in terms of π. It is related to degree measure by the following conversion factor: π radians $= 180°$.
- To convert from degree measure to radians, multiply by $\frac{\pi}{180°}$.
- To convert from radians to degrees, multiply by $\frac{180°}{\pi}$.

CHECK Your Understanding

1. A point is rotated about a circle of radius 1. Its start and finish are shown. State the rotation in radian measure and in degree measure.



Communication *Tip*

Recall that counterclockwise rotation is represented using positive angles, while clockwise rotation is represented using **negative angles**.

2. Sketch each rotation about a circle of radius 1.

a) π **c)** $\frac{2\pi}{3}$ **e)** $\frac{5\pi}{3}$ **g)** $-\frac{\pi}{2}$

b) $\frac{\pi}{3}$ **d)** $\frac{4\pi}{3}$ **f)** $-\pi$ **h)** $-\frac{\pi}{4}$

3. Convert each angle from degrees to radians, in exact form.

a) $75°$ **b)** $200°$ **c)** $400°$ **d)** $320°$

4. Convert each angle from radians to degrees. Express the measure correct to two decimal places, if necessary.

a) $\frac{5\pi}{3}$ **b)** 0.3π **c)** 3 **d)** $\frac{11\pi}{4}$

PRACTISING

5. **a)** Determine the measure of the central angle that is formed by an arc length of 5 cm in a circle with a radius of 2.5 cm. Express the measure in both radians and degrees, correct to one decimal place.

b) Determine the arc length of the circle in part a) if the central angle is $200°$.

6. Determine the arc length of a circle with a radius of 8 cm if

a) the central angle is 3.5

b) the central angle is $300°$

7. Convert to radian measure.

K **a)** $90°$ **c)** $-180°$ **e)** $-135°$ **g)** $240°$

b) $270°$ **d)** $45°$ **f)** $60°$ **h)** $-120°$

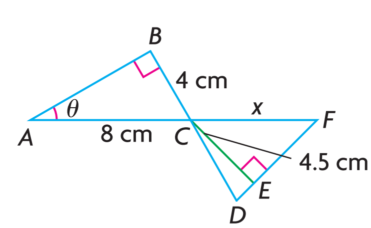
8. Convert to degree measure.

a) $\frac{2\pi}{3}$ **c)** $\frac{\pi}{4}$ **e)** $\frac{7\pi}{6}$ **g)** $\frac{11\pi}{6}$

b) $-\frac{5\pi}{3}$ **d)** $-\frac{3\pi}{4}$ **f)** $-\frac{3\pi}{2}$ **h)** $-\frac{9\pi}{2}$

9. If a circle has a radius of 65 m, determine the arc length for each of the following central angles.

a) $\frac{19\pi}{20}$ **b)** 1.25 **c)** $150°$



10. Given $\angle DCE = \frac{\pi}{12}$ radians and $CE = 4.5$ cm, determine the size of
A θ and x .

11. A wind turbine has three blades, each measuring 3 m from centre to tip. At a particular time, the turbine is rotating four times a minute.

a) Determine the angular velocity of the turbine in radians/second.

b) How far has the tip of a blade travelled after 5 min?

12. A wheel is rotating at an angular velocity of 1.2π radians/s, while a point on the circumference of the wheel travels 9.6π m in 10 s.

a) How many revolutions does the wheel make in 1 min?

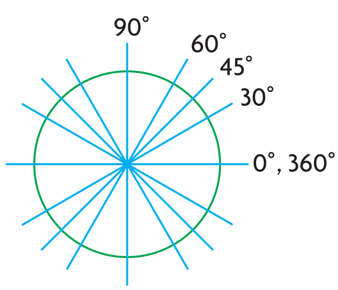
b) What is the radius of the wheel?

13. T Two pieces of mud are stuck to the spoke of a bicycle wheel. Piece A is closer to the circumference of the tire, while piece B is closer to the centre of the wheel.

a) Is the angular velocity at which piece A is travelling greater than, less than, or equal to the angular velocity at which piece B is travelling?

b) Is the velocity at which piece A is travelling greater than, less than, or equal to the velocity at which piece B is travelling?

c) If the angular velocity of the bicycle wheel increased, would the velocity at which piece A is travelling as a percent of the velocity at which piece B is travelling increase, decrease, or stay the same?



14. C In your notebook, sketch the diagram shown and label each angle, in degrees, for one revolution. Then express each of these angles in exact radian measure.

Extending

15. Circle A has a radius of 15 cm and a central angle of $\frac{\pi}{6}$ radians, circle B has a radius of 17 cm and a central angle of $\frac{\pi}{7}$ radians, and circle C has a radius of 14 cm and a central angle of $\frac{\pi}{5}$ radians. Put the circles in order, from smallest to largest, based on the lengths of the arcs subtending the central angles.

16. The members of a high-school basketball team are driving from Calgary to Vancouver, which is a distance of 675 km. Each tire on their van has a radius of 32 cm. If the team members drive at a constant speed and cover the distance from Calgary to Vancouver in 6 h 45 min, what is the angular velocity, in radians/second, of each tire during the drive?