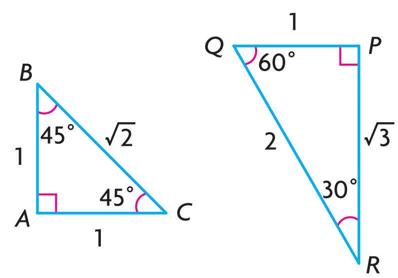
6.2 Radian Measure and Angles on the Cartesian Plane

GOAL

Use the Cartesian plane to evaluate the trigonometric ratios for angles between 0 and 2π.

LEARN ABOUT the Math

Recall that the special triangles shown can be used to determine the exact values of the primary and reciprocal trigonometric ratios for some angles measured in degrees.



? How can these special triangles be used to determine the exact values of the trigonometric ratios for angles expressed in radians?

EXAMPLE 1 Connecting radians and the special triangles

Determine the radian measures of the angles in the special triangles, and calculate their primary trigonometric ratios.

Solution

$\angle Q = 60°$

$60° = \overset{1}{\cancel{60°}}\left(\dfrac{\pi}{\underset{3}{\cancel{180°}}}\right) = \dfrac{\pi}{3}$

$\angle R = 30°$

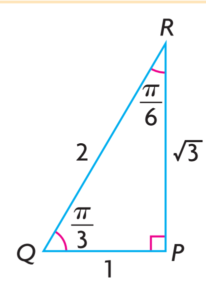
$30° = \overset{1}{\cancel{30°}}\left(\dfrac{\pi}{\underset{6}{\cancel{180°}}}\right) = \dfrac{\pi}{6}$

$\angle B = \angle C = 45°$

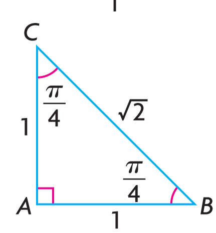
$45° = \overset{1}{\cancel{45°}}\left(\dfrac{\pi}{\underset{4}{\cancel{180°}}}\right) = \dfrac{\pi}{4}$

$\angle P = \angle A = 90°$

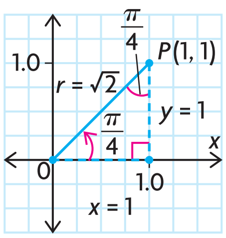
$90° = \overset{1}{\cancel{90°}}\left(\dfrac{\pi}{\underset{2}{\cancel{180°}}}\right) = \dfrac{\pi}{2}$



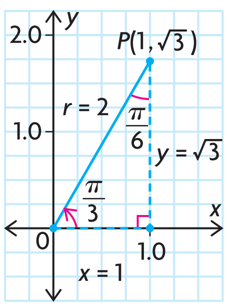
$\triangle PQR$ is the 30°, 60°, 90° special triangle. Multiply each angle by $\frac{\pi}{180°}$ to convert from degrees to radians.



$\triangle ABC$ is the 45°, 45°, 90° special triangle. Multiply each angle by $\frac{\pi}{180°}$ to convert from degrees to radians.



$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \csc\frac{\pi}{4} = \sqrt{2}$$
$$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \sec\frac{\pi}{4} = \sqrt{2}$$
$$\tan\frac{\pi}{4} = 1 \qquad \cot\frac{\pi}{4} = 1$$



$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} \qquad \csc\frac{\pi}{3} = \frac{2}{\sqrt{3}}$$
$$\cos\frac{\pi}{3} = \frac{1}{2} \qquad \sec\frac{\pi}{3} = 2$$
$$\tan\frac{\pi}{3} = \sqrt{3} \qquad \cot\frac{\pi}{3} = \frac{1}{\sqrt{3}}$$

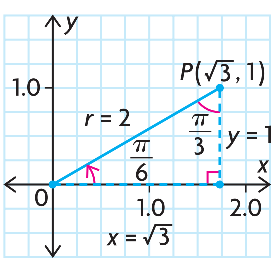
Draw each special angle on the Cartesian plane in **standard position**. Use the trigonometric definitions of angles on the Cartesian plane to determine the exact value of each angle. Recall that

$\sin\theta = \frac{y}{r}$ $\csc\theta = \frac{r}{y}$

$\cos\theta = \frac{x}{r}$ $\sec\theta = \frac{r}{x}$

$\tan\theta = \frac{y}{x}$ $\cot\theta = \frac{x}{y}$

where $x^2 + y^2 = r^2$ and $r > 0$.



$$\sin\frac{\pi}{6} = \frac{1}{2} \qquad \csc\frac{\pi}{6} = 2$$
$$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \qquad \sec\frac{\pi}{6} = \frac{2}{\sqrt{3}}$$
$$\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} \qquad \cot\frac{\pi}{6} = \sqrt{3}$$

Reflecting

A. Compare the exact values of the trigonometric ratios in each special triangle when the angles are given in radians and when the angles are given in degrees.

B. Explain why the strategy that is used to determine the value of a trigonometric ratio for a given angle on the Cartesian plane is the same when the angle is expressed in radians and when the angle is expressed in degrees.

APPLY the Math

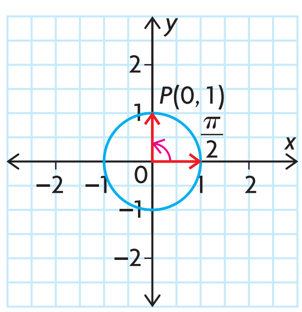
EXAMPLE 2 Selecting a strategy to determine the exact value of a trigonometric ratio

Determine the exact value of each trigonometric ratio.

a) $\sin\left(\frac{\pi}{2}\right)$ **b)** $\cot\left(\frac{3\pi}{2}\right)$

Solution

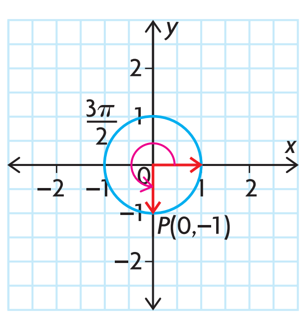
a)



$\frac{\pi}{2}$ is one-quarter of a full revolution, and the point $P(0, 1)$ lies on the unit circle, as shown. Draw the angle in standard position with its terminal arm on the positive y-axis. From the drawing, $x = 0$, $y = 1$, and $r = 1$.

$$\sin\left(\frac{\pi}{2}\right) = \frac{y}{r} = \frac{1}{1} = 1$$

b)



$\frac{3\pi}{2}$ is three-quarters of a full revolution, and the point $P(0, -1)$ lies on the unit circle, as shown. Draw the angle in standard position with its terminal arm on the negative y-axis. From the drawing, $x = 0$, $y = -1$ and $r = 1$.

$$\cot\left(\frac{3\pi}{2}\right) = \frac{x}{y} = \frac{0}{-1} = 0$$

The relationships between the principal angle, its related acute angle, and the trigonometric ratios for angles in standard position are the same when the angles are measured in radians and degrees.

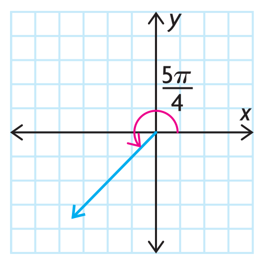
EXAMPLE 3 Selecting a strategy to determine the exact value of a trigonometric ratio

Determine the exact value of each trigonometric ratio.

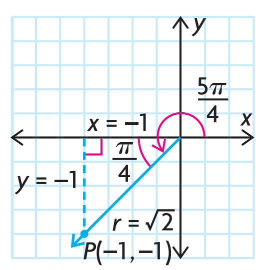
a) $\cos\left(\frac{5\pi}{4}\right)$ **b)** $\csc\left(\frac{11\pi}{6}\right)$

Solution A: Using the special angles

a)



Sketch the angle in standard position. π is a half of a revolution. $\frac{5\pi}{4}$ is halfway between π and $\frac{3\pi}{2}$, and lies in the third quadrant with a related angle of $\frac{5\pi}{4} - \pi$, or $\frac{\pi}{4}$.

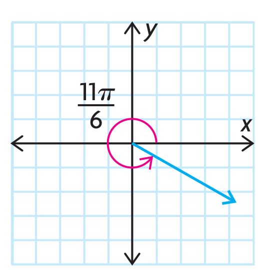


$$\cos\left(\frac{5\pi}{4}\right) = \frac{x}{r} = \frac{-1}{\sqrt{2}}$$

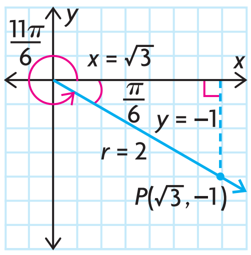
$\frac{\pi}{4}$ is in the 1, 1, $\sqrt{2}$ special triangle. Position this triangle so the right angle lies on the negative x-axis.

Since $(-1, -1)$ lies on the terminal arm, $x = -1$, $y = -1$, and $r = \sqrt{2}$. Therefore, the cosine ratio has a negative value.

b)



Sketch the angle in standard position. $\frac{11\pi}{6}$ is between $\frac{3\pi}{2}$ and 2π, and lies in the fourth quadrant with a related angle of $2\pi - \frac{11\pi}{6}$, or $\frac{\pi}{6}$.

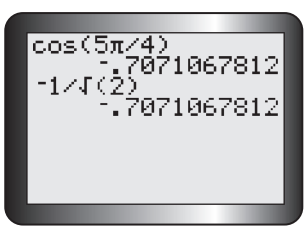


$\frac{\pi}{6}$ is in the 1, $\sqrt{3}$, 2 special triangle. Position it so that the right angle lies on the positive *x*-axis. Since the point $(\sqrt{3}, -1)$ lies on the terminal arm, $x = \sqrt{3}$, $y = -1$, and $r = 2$. Therefore, the csc ratio has a negative value.

$$\csc\left(\frac{11\pi}{6}\right) = \frac{r}{y}$$
$$= \frac{2}{-1} = -2$$

Solution B: Using a calculator

a)

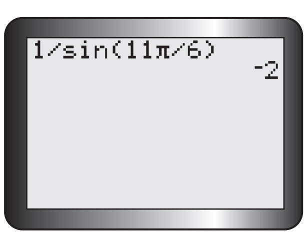


Set the calculator to radian mode. Enter the expression.

The result is a decimal. Entering $-\frac{1}{\sqrt{2}}$ confirms that the answer is equivalent to this decimal.

$$\cos\left(\frac{5\pi}{4}\right) = \frac{-1}{\sqrt{2}}$$

b)

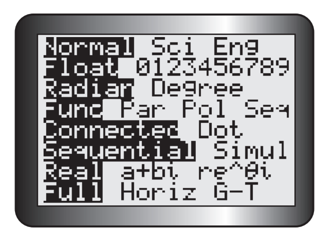


There is no csc key on the calculator. Use the fact that cosecant is the reciprocal of sine.

$$\csc\left(\frac{11\pi}{6}\right) = -2$$

Tech | *Support*

To put a graphing calculator in radian mode, press the MODE key, scroll to Radian, and press ENTER.



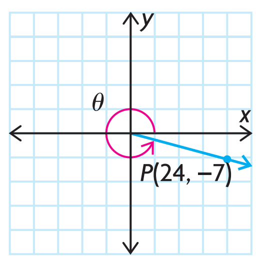
EXAMPLE 4 Solving a trigonometric equation that involves radians

If $\tan\theta = -\frac{7}{24}$, where $0 \le \theta \le 2\pi$, evaluate θ to the nearest hundredth.

Solution

$$\tan\theta = -\frac{7}{24} = \frac{y}{x}$$

There are two possibilities to consider:
$x = 24$, $y = -7$ and
$x = -24$, $y = 7$.

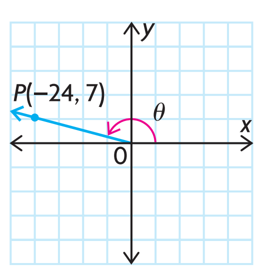


For the ordered pair $(24, -7)$, the terminal arm of the angle θ lies in the fourth quadrant.
$\frac{3\pi}{2} < \theta < 2\pi$

Use a calculator to determine the related acute angle by calculating the inverse tan of $\frac{7}{24}$. The related angle is 0.28, rounded to two decimal places. Subtract 0.28 from 2π to determine one measure of θ.

$2\pi - 0.28 \doteq 6.00$

In the fourth quadrant, θ is about 6.00.



For the ordered pair $(-24, 7)$, the terminal arm of θ lies in the second quadrant, $\frac{\pi}{2} < \theta < \pi$, and also has a related angle of 0.28. Subtract 0.28 from π to determine the other measure of θ.

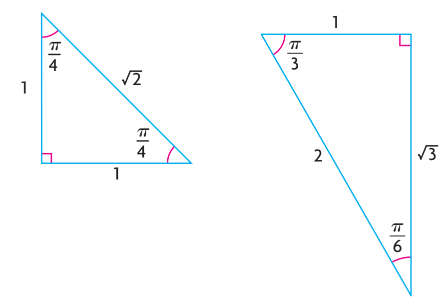
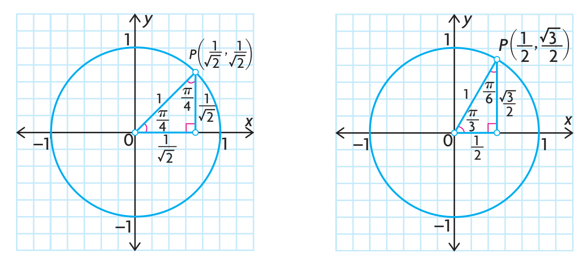
$\pi - 0.28 \doteq 2.86$

In the second quadrant, θ is about 2.86.

In Summary

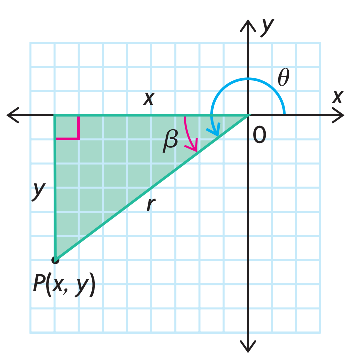
Key Ideas

- The angles in the special triangles can be expressed in radians, as well as in degrees. The radian measures can be used to determine the exact values of the trigonometric ratios for multiples of these angles between 0 and 2π.
- The strategies that are used to determine the values of the trigonometric ratios when an angle is expressed in degrees on the Cartesian plane can also be used when the angle is expressed in radians.

The Special Triangles	The Special Triangles on the Cartesian Plane Using a Circle of Radius 1
	

Need to Know

- The trigonometric ratios for any principal angle, θ, in standard position can be determined by finding the related acute angle, β, using coordinates of any point that lies on the terminal arm of the angle.

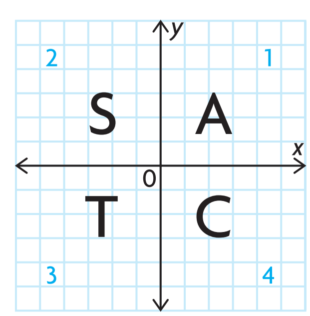


From the Pythagorean theorem, $r^2 = x^2 + y^2$, if $r > 0$.

$\sin\theta = \frac{y}{r}$ $\quad \cos\theta = \frac{x}{r}$ $\quad \tan\theta = \frac{y}{x}$

$\csc\theta = \frac{r}{y}$ $\quad \sec\theta = \frac{r}{x}$ $\quad \cot\theta = \frac{x}{y}$

- The CAST rule is an easy way to remember which primary trigonometric ratios are positive in which quadrant. Since r is always positive, the sign of each primary ratio depends on the signs of the coordinates of the point.
  - In quadrant 1, **A**ll (A) ratios are positive because both x and y are positive.
  - In quadrant 2, only **S**ine (S) is positive, since x is negative and y is positive.
  - In quadrant 3, only **T**angent (T) is positive because both x and y are negative.
  - In quadrant 4, only **C**osine (C) is positive, since x is positive and y is negative.



CHECK *Your Understanding*

1. For each trigonometric ratio, use a sketch to determine in which quadrant the terminal arm of the principal angle lies, the value of the related acute angle, and the sign of the ratio.

a) $\sin \frac{3\pi}{4}$

b) $\cos \frac{5\pi}{3}$

c) $\tan \frac{4\pi}{3}$

d) $\sec \frac{5\pi}{6}$

e) $\cos \frac{2\pi}{3}$

f) $\cot \frac{7\pi}{4}$

2. Each of the following points lies on the terminal arm of an angle in standard position.

i) Sketch each angle.

ii) Determine the value of r.

iii) Determine the primary trigonometric ratios for the angle.

iv) Calculate the radian value of θ, to the nearest hundredth, where $0 \leq \theta \leq 2\pi$.

a) $(6, 8)$

b) $(-12, -5)$

c) $(4, -3)$

d) $(0, 5)$

3. Determine the primary trigonometric ratios for each angle.

a) $-\frac{\pi}{2}$

b) $-\pi$

c) $\frac{7\pi}{4}$

d) $-\frac{\pi}{6}$

4. State an equivalent expression in terms of the related acute angle.

a) $\sin \frac{5\pi}{6}$

b) $\cos \frac{5\pi}{3}$

c) $\cot \left(-\frac{\pi}{4}\right)$

d) $\sec \frac{7\pi}{6}$

PRACTISING

5. Determine the exact value of each trigonometric ratio.
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a) $\sin \frac{2\pi}{3}$

b) $\cos \frac{5\pi}{4}$

c) $\tan \frac{11\pi}{6}$

d) $\sin \frac{7\pi}{4}$

e) $\csc \frac{5\pi}{6}$

f) $\sec \frac{5\pi}{3}$

6. For each of the following values of $\cos \theta$, determine the radian value of θ if $\pi \leq \theta \leq 2\pi$.

a) $-\frac{1}{2}$ **c)** $-\frac{\sqrt{2}}{2}$ **e)** 0

b) $\frac{\sqrt{3}}{2}$ **d)** $-\frac{\sqrt{3}}{2}$ **f)** -1

7. The terminal arm of an angle in standard position passes through each of the following points. Find the radian value of the angle in the interval $[0, 2\pi]$, to the nearest hundredth.

a) $(-7, 8)$ **c)** $(3, 11)$ **e)** $(9, 10)$

b) $(12, 2)$ **d)** $(-4, -2)$ **f)** $(6, -1)$

8. State an equivalent expression in terms of the related acute angle.

a) $\cos \frac{3\pi}{4}$ **c)** $\csc \left(-\frac{\pi}{3}\right)$ **e)** $\sin \frac{-\pi}{6}$

b) $\tan \frac{11\pi}{6}$ **d)** $\cot \frac{2\pi}{3}$ **f)** $\sec \frac{7\pi}{4}$

9. **A** A leaning flagpole, 5 m long, makes an obtuse angle with the ground. If the distance from the tip of the flagpole to the ground is 3.4 m, determine the radian measure of the obtuse angle, to the nearest hundredth.

10. The needle of a compass makes an angle of 4 radians with the line pointing east from the centre of the compass. The tip of the needle is 4.2 cm below the line pointing west from the centre of the compass. How long is the needle, to the nearest hundredth of a centimetre?

11. **T** A clock is showing the time as exactly 3:00 p.m. and 25 s. Because a full minute has not passed since 3:00, the hour hand is pointing directly at the 3 and the minute hand is pointing directly at the 12. If the tip of the second hand is directly below the tip of the hour hand, and if the length of the second hand is 9 cm, what is the length of the hour hand?

12. **C** If you are given an angle, θ, that lies in the interval $\theta \in \left[\frac{\pi}{2}, 2\pi\right]$, how would you determine the values of the primary trigonometric ratios for this angle?

13. You are given $\cos \theta = -\frac{5}{13}$, where $0 \leq \theta \leq 2\pi$.

a) In which quadrant(s) could the terminal arm of θ lie?

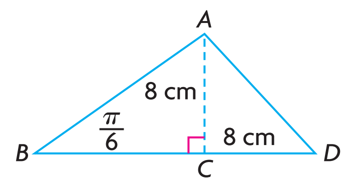
b) Determine all the possible trigonometric ratios for θ.

c) State all the possible radian values of θ, to the nearest hundredth.

14. Use special triangles to show that the equation $\cos\left(\frac{5\pi}{6}\right) = \cos(-150°)$ is true.

15. Show that $2\sin^2\theta - 1 = \sin^2\theta - \cos^2\theta$ for $\frac{11\pi}{6}$.

16. Determine the length of AB. Find the sine, cosine, and tangent ratios of $\angle D$, given $AC = CD = 8$ cm.



17. Given that x is an acute angle, draw a diagram of both angles (in standard position) in each of the following equalities. For each angle, indicate the related acute angle as well as the principal angle. Then, referring to your drawings, explain why each equality is true.

a) $\sin x = \sin(\pi - x)$

b) $\sin x = -\sin(2\pi - x)$

c) $\cos x = -\cos(\pi - x)$

d) $\tan x = \tan(\pi + x)$

Extending

18. Find the sine of the angle formed by two rays that start at the origin of the Cartesian plane if one ray passes through the point $(3\sqrt{3}, 3)$ and the other ray passes through the point $(-4, 4\sqrt{3})$. Round your answer to the nearest hundredth, if necessary.

19. Find the cosine of the angle formed by two rays that start at the origin of the Cartesian plane if one ray passes through the point $(6\sqrt{2}, 6\sqrt{2})$ and the other ray passes through the point $(-7\sqrt{3}, 7)$. Round your answer to the nearest hundredth, if necessary.

20. Julie noticed that the ranges of the sine and cosine functions go from -1 to 1, inclusive. She then began to wonder about the reciprocals of these functions—that is, the cosecant and secant functions. What do you think the ranges of these functions are? Why?

21. The terminal arm of θ is in the fourth quadrant. If $\cot\theta = -\sqrt{3}$, then calculate $\sin\theta\cot\theta - \cos^2\theta$.