6.4 Transformations of Trigonometric Functions

GOAL

Use transformations to sketch the graphs of the primary trigonometric functions in radians.

YOU WILL NEED

- graph paper
- graphing calculator

LEARN ABOUT the Math

The following transformations are applied to the graph of $y = \sin x$, where $0 \le x \le 2\pi$:

- a vertical stretch by a factor of 3
- a horizontal compression by a factor of $\frac{1}{2}$
- a horizontal translation $\frac{\pi}{6}$ to the left
- a vertical translation 1 down

? What is the equation of the transformed function, and what does its graph look like?

EXAMPLE 1 Selecting a strategy to apply transformations and graph a sine function

Use the transformations above to sketch the graph of the transformed function in the interval $0 \le x \le 2\pi$.

Solution A: Applying the transformation to the key points of the parent function

$y = \sin x$ is the parent function.

x	$y = \sin(x)$
0	0
$\frac{\pi}{2}$	1
π	0
$\frac{3\pi}{2}$	-1
2π	0

One cycle of the parent function can be described with five key points. By applying the relevant transformations to these points, a complete cycle of the transformed function can be graphed.

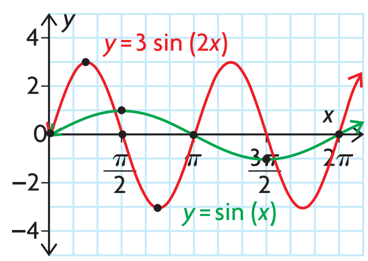
$y = 3\sin\left(2\left(x + \frac{\pi}{6}\right)\right) - 1$ is the equation of the transformed function.

Recall that, in the general function $y = af(k(x - d)) + c$, each parameter is associated with a specific transformation. In this case,
$a = 3$ (vertical stretch)
$k = \frac{1}{\frac{1}{2}} = 2$ (horizontal compression)
$d = -\frac{\pi}{6}$ (translation left)
$c = -1$ (translation down)

$(x, y) \rightarrow \left(\frac{1}{2}x, 3y\right)$

Parent Function, $y = \sin x$	Stretched/Compressed Function, $y = 3 \sin (2x)$
$(0, 0)$	$\left(\frac{1}{2}(0), 3(0)\right) = (0, 0)$
$\left(\frac{\pi}{2}, 1\right)$	$\left(\frac{1}{2}\left(\frac{\pi}{2}\right), 3(1)\right) = \left(\frac{\pi}{4}, 3\right)$
$(\pi, 0)$	$\left(\frac{1}{2}(\pi), 3(0)\right) = \left(\frac{\pi}{2}, 0\right)$
$\left(\frac{3\pi}{2}, -1\right)$	$\left(\frac{1}{2}\left(\frac{3\pi}{2}\right), 3(-1)\right) = \left(\frac{3\pi}{4}, -3\right)$
$(2\pi, 0)$	$\left(\frac{1}{2}(2\pi), 3(0)\right) = (\pi, 0)$

The parameters k and d affect the x-coordinates of each point on the parent function, and the parameters a and c affect the y-coordinates. All stretches/compressions and reflections must be applied before any translations. In this example, each x-coordinate of the five key points is multiplied by $\frac{1}{2}$, and each y-coordinate is multiplied by 3.



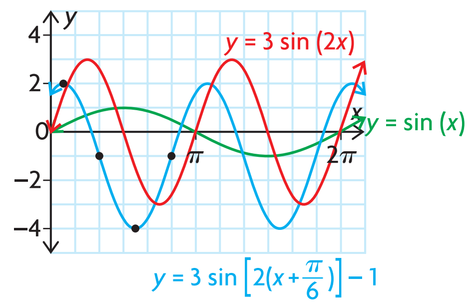
Plot the key points of the parent function and the key points of the transformed function, and draw smooth curves through them. Extend the red curve for one more cycle.

$\left(\frac{1}{2}x, 3y\right) \rightarrow \left(\frac{1}{2}x - \frac{\pi}{6}, 3y - 1\right)$

Stretched/Compressed Function, $y = 3 \sin (2x)$	Final Transformed Function, $y = 3 \sin \left(2\left(x + \frac{\pi}{6}\right)\right) - 1$
$(0, 0)$	$\left(0 - \frac{\pi}{6}, 0 - 1\right) = \left(-\frac{\pi}{6}, -1\right)$
$\left(\frac{\pi}{4}, 3\right)$	$\left(\frac{\pi}{4} - \frac{\pi}{6}, 3 - 1\right) = \left(\frac{\pi}{12}, 2\right)$
$\left(\frac{\pi}{2}, 0\right)$	$\left(\frac{\pi}{2} - \frac{\pi}{6}, 0 - 1\right) = \left(\frac{\pi}{3}, -1\right)$
$\left(\frac{3\pi}{4}, -3\right)$	$\left(\frac{3\pi}{4} - \frac{\pi}{6}, -3 - 1\right) = \left(\frac{7\pi}{12}, -4\right)$
$(\pi, 0)$	$\left(\pi - \frac{\pi}{6}, 0 - 1\right) = \left(\frac{5\pi}{6}, -1\right)$

Each x-coordinate of the key points on the previous function now has $\frac{\pi}{6}$ subtracted from it, and each y-coordinate has 1 subtracted from it.

These five points represent one complete cycle of the graph. To extend the graph to 2π, copy this cycle by adding the period of π to each x-coordinate in the table of the transformed key points.



Plot the key points of the final transformed function, and draw a smooth curve through them.

Note that the vertical stretch and translation cause corresponding changes in the range of the parent function. The range of the parent function is $-1 \leq y \leq 1$, and the range of the transformed function is $-4 \leq y \leq 2$.

Solution B: Using the features of the transformed function

$y = 3 \sin \left(2\left(x + \frac{\pi}{6}\right)\right) - 1$ is the equation of the transformed function. It has the following characteristics:

Amplitude $= 3$

Period $= \frac{2\pi}{2} = \pi$

Equation of the axis: $y = -1$

Recall that each parameter in the general function $y = af(k(x - d)) + c$ is associated with a specific transformation. For the transformations applied to $f(x) = \sin x$,

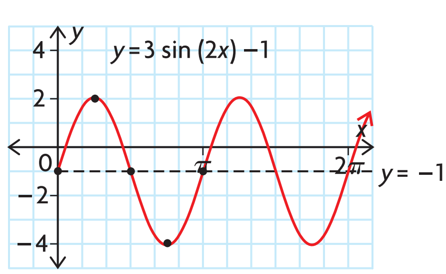
$a = 3$ (vertical stretch)

$k = \frac{1}{\frac{1}{2}} = 2$ (horizontal compression)

$d = -\frac{\pi}{6}$ (translation left)

$c = -1$ (translation down)

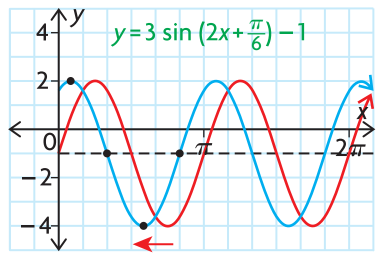
Sketch the graph of $y = 3 \sin (2x) - 1$ by plotting its axis, points on its axis, and maximum and minimum values.



Since the axis is $y = -1$ and the amplitude is 3, the graph has a maximum at 2 and a minimum at -4. Since this is a sine function with a period of π, the maximum occurs at $x = \frac{\pi}{4}$, and the minimum occurs at $x = \frac{3\pi}{4}$. The graph has points on the axis when $x = 0$, $x = \frac{\pi}{2}$, and $x = \pi$.

Since the given domain is $0 \leq \theta \leq 2\pi$, add the period π to each point that was plotted for the first cycle and draw a smooth curve.

$y = 3 \sin \left(2\left(x + \frac{\pi}{6}\right)\right) - 1$ is the function $y = 3 \sin (2x) - 1$ translated $\frac{\pi}{6}$ to the left.



Apply the horizontal translation to the previous graph by shifting the maximum and minimum points and the points on the axis $\frac{\pi}{6}$ to the left.

Reflecting

A. What transformations affect each of the following characteristics of a sinusoidal function?

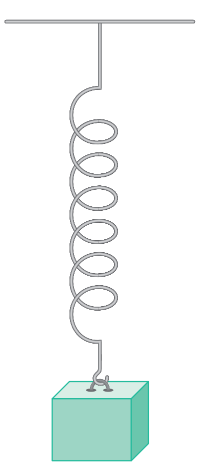
i) period **ii)** amplitude **iii)** equation of the axis

B. In both solutions, it was necessary to extend the graphs after the final transformed points were plotted. Explain how this was done.

C. Which strategy for graphing sinusoidal functions do you prefer? Explain why.

APPLY the Math

EXAMPLE 2 Using the graph of a sinusoidal function to solve a problem



A mass on a spring is pulled toward the floor and released, causing it to move up and down. Its height, in centimetres, above the floor after t seconds is given by the function $h(t) = 10 \sin (2\pi t + 1.5\pi) + 15$, $0 \leq t \leq 3$. Sketch a graph of height versus time. Then use your graph to predict when the mass will be 18 cm above the floor as it travels in an upward direction.

Solution

$h(t) = 10 \sin (2\pi t + 1.5\pi) + 15$

$h(t) = 10 \sin (2\pi(t + 0.75)) + 15$

For this function, the amplitude is 10 and the period is 1. The equation of the axis is $h = 15$. The function undergoes a horizontal translation 0.75 to the left.

Determine the characteristics that define the graph of this function. To do so, divide out the common factor from the argument. Then determine the values of the parameters a, k, d, and c.

$a = 10$

$k = 2\pi$, so the period is $\frac{2\pi}{2\pi} = 1$

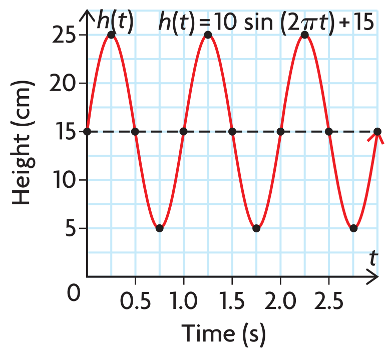
$d = -0.75$

$c = 15$

argument

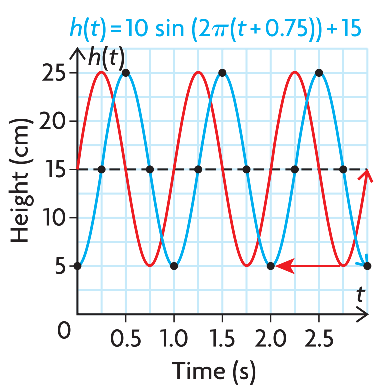
the expression on which a function operates; in Example 2, sin is the function and it operates on the expression $2\pi t + 1.5\pi$; so $2\pi t + 1.5\pi$ is the argument

Sketch the graph of $h(t) = 10 \sin (2\pi t) + 15$ over one cycle using the axis, amplitude, and period.



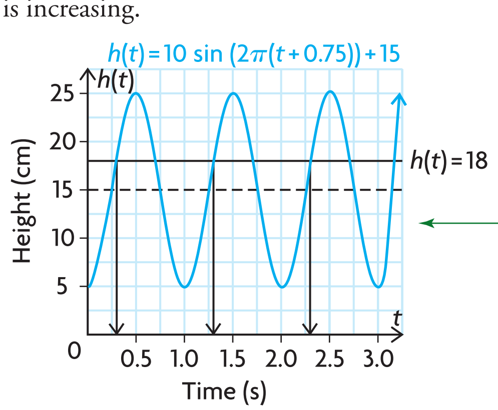
Since the axis is $h(t) = 15$ and the amplitude is 10, the graph will have a maximum at 25 and a minimum at 5. Since this is a sine function with a period of 1, these points will occur at $t = \frac{1}{4}$ and $t = \frac{3}{4}$. The graph has points on the axis when $t = 0$, $t = \frac{1}{2}$, and $t = 1$.

Since the given domain is $0 \leq t \leq 3$, add the period 1 to each point that was plotted for the first cycle. Repeat using the points on the second cycle to get three complete cycles. Then draw a smooth curve.



Apply the horizontal translation to the previous graph by shifting the maximum and minimum points and the points on the axis 0.75 to the left.

The spring is on its way up on the parts of the graph where the height is increasing.



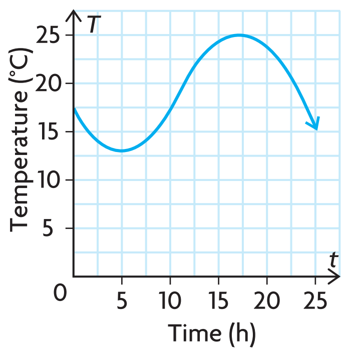
Use the graph to estimate when the spring will be 18 cm above the floor on the intervals $t \in [0, 0.5]$, $[1.0, 1.5]$, and $[2.0, 2.5]$.

On its way up, the spring is at a height of 18 cm at about 0.3 s, 1.3 s, and 2.3 s.

If you are given a graph of a sinusoidal function, then characteristics of its graph can be used to determine the equation of the function.

EXAMPLE 3 Connecting the features of the graph of a sinusoidal function to its equation

The following graph shows the temperature in Nellie's dorm room over a 24 h period.



Determine the equation of this sinusoidal function.

Solution

Use the graph to determine the values of the parameters a, k, d, and c, and write the equation.

The graph resembles the cosine function, so its equation is of the form $y = a\cos(k(x - d)) + c$.

The axis is $c = \frac{13 + 25}{2} = 19$.

The value of c indicates the horizontal axis of the function. The horizontal axis is the mean of the maximum and minimum values.

$a = \frac{25 - 13}{2} = 6$

The value of a indicates the amplitude of the function. The amplitude is half the difference between the maximum and minimum values.

$$\text{Period} = \frac{2\pi}{k}, \text{ so } 24 = \frac{2\pi}{k}$$

$$24k = 2\pi$$

$$k = \frac{\pi}{12}$$

The value of k is related to the period of the function.

If you assume that this cycle repeats itself over several days, then the period is 1 day, or 24 h.

$d = 17$

The equation is $T(t) = 6\cos\left(\frac{\pi}{12}(t - 17)\right) + 19$.

Let us use a cosine function. The parent function has a maximum value at $t = 0$.

This graph has a maximum value at $t = 17$. Therefore, we translate the function 17 units to the right.

In Summary

Key Idea

- The graphs of functions of the form $f(x) = a \sin (k(x - d)) + c$ and $f(x) = a \cos (k(x - d)) + c$ are transformations of the parent functions $y = \sin (x)$ and $y = \cos (x)$, respectively.

  To sketch these functions, you can use a variety of strategies. Two of these strategies are given below:

  1. Begin with the key points in one cycle of the parent function and apply any stretches/compressions and reflections to these points: $(x, y) \rightarrow \left(\frac{x}{k}, ay\right)$. Take each of the new points, and apply any translations: $\left(\frac{x}{k}, ay\right) \rightarrow \left(\frac{x}{k} + d, ay + c\right)$.
     To graph more cycles, as required by the given domain, add multiples of the period to the x-coordinates of these transformed points and draw a smooth curve.
  2. Using the given equation, determine the equation of the axis, amplitude, and period of the function. Use this information to determine the location of the maximum and minimum points and the points that lie on the axis for one cycle. Plot these points, and then apply the horizontal translation to these points. To graph more cycles, as required by the domain, add multiples of the period to the x-coordinates of these points and draw a smooth curve.

Need to Know

- The parameters in the equations $f(x) = a \sin (k(x - d)) + c$ and $f(x) = a \cos (k(x - d)) + c$ give useful information about transformations and characteristics of the function.

Transformations of the Parent Function	Characteristics of the Transformed Function
$\lvert a\rvert$ gives the vertical stretch/compression factor. If $a < 0$, there is also a reflection in the x-axis.	$\lvert a\rvert$ gives the amplitude.
$\left\lvert\frac{1}{k}\right\rvert$ gives the horizontal stretch/compression factor. If $k < 0$, there is also a reflection in the y-axis.	$\frac{2\pi}{\lvert k\rvert}$ gives the period.
d gives the horizontal translation.	d gives the horizontal translation.
c gives the vertical translation.	$y = c$ gives the equation of the axis.

- If the independent variable has a coefficient other than $+1$, the argument must be factored to separate the values of k and c. For example,
  $y = 3 \cos (2x + \pi)$ should be changed to $y = 3 \cos \left(2\left(x + \frac{\pi}{2}\right)\right)$.

CHECK Your Understanding

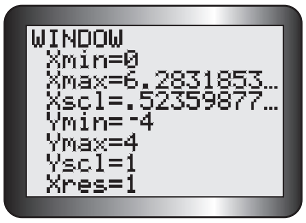
1. State the period, amplitude, horizontal translation, and equation of the axis for each of the following trigonometric functions.

a) $y = 0.5 \cos (4x)$

b) $y = \sin \left(x - \frac{\pi}{4}\right) + 3$

c) $y = 2 \sin (3x) - 1$

d) $y = 5 \cos \left(-2x + \frac{\pi}{3}\right) - 2$



2. Suppose the trigonometric functions in question 1 are graphed using a graphing calculator in radian mode and the window settings shown. Which functions produce a graph that is not cut off on the top or bottom and that displays at least one cycle?

3. Identify the key characteristics of $y = -2\cos(4x + \pi) + 4$, and sketch its graph. Check your graph with a graphing calculator.

PRACTISING

4. The following trigonometric functions have the parent function $f(x) = \sin x$. They have undergone no horizontal translations and no reflections in either axis. Determine the equation of each function.

a) The graph of this trigonometric function has a period of π and an amplitude of 25. The equation of the axis is $y = -4$.

b) The graph of this trigonometric function has a period of 10 and an amplitude of $\frac{2}{5}$. The equation of the axis is $y = \frac{1}{15}$.

c) The graph of this trigonometric function has a period of 6π and an amplitude of 80. The equation of the axis is $y = -\frac{9}{10}$.

d) The graph of this trigonometric function has a period of $\frac{1}{2}$ and an amplitude of 11. The equation of the axis is $y = 0$.

5. K State the period, amplitude, and equation of the axis of the trigonometric function that produces each of the following tables of values. Then use this information to write the equation of the function.

a)

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
y	0	18	0	-18	0

b)

x	0	π	2π	3π	4π
y	-2	4	-2	-8	-2

c)

x	0	3π	6π	9π	12π
y	4	9	4	9	4

d)

x	0	2π	4π	6π	8π
y	-3	1	-3	1	-3

6. State the transformations that were applied to the parent function $f(x) = \sin x$ to obtain each of the following transformed functions. Then graph the transformed functions.

a) $f(x) = 4 \sin x + 3$

b) $f(x) = -\sin\left(\frac{1}{4}x\right)$

c) $f(x) = \sin(x - \pi) - 1$

d) $f(x) = \sin\left(4x + \frac{2\pi}{3}\right)$

7. The trigonometric function $f(x) = \cos x$ has undergone the following sets of transformations. For each set of transformations, determine the equation of the resulting function and sketch its graph.

a) vertical compression by a factor of $\frac{1}{2}$, vertical translation 3 units up

b) horizontal stretch by a factor of 2, reflection in the y-axis

c) vertical stretch by a factor of 3, horizontal translation $\frac{\pi}{2}$ to the right

d) horizontal compression by a factor of $\frac{1}{2}$, horizontal translation $\frac{\pi}{2}$ to the left

8. Sketch each graph for $0 \le x \le 2\pi$. Verify your sketch using graphing technology.

a) $y = 3 \sin\left(2\left(x - \frac{\pi}{6}\right)\right) + 1$

b) $y = 5 \cos\left(x + \frac{\pi}{4}\right) - 2$

c) $y = -2 \sin\left(2\left(x + \frac{\pi}{4}\right)\right) + 2$

d) $y = -\cos\left(0.5x - \frac{\pi}{6}\right) + 3$

e) $y = 0.5 \sin\left(\frac{x}{4} - \frac{\pi}{16}\right) - 5$

f) $y = \frac{1}{2} \cos\left(\frac{x}{2} - \frac{\pi}{12}\right) - 3$

9. A Each person's blood pressure is different, but there is a range of blood pressure values that is considered healthy. The function $P(t) = -20 \cos \frac{5\pi}{3}t + 100$ models the blood pressure, p, in millimetres of mercury, at time t, in seconds, of a person at rest.

a) What is the period of the function? What does the period represent for an individual?

b) How many times does this person's heart beat each minute?

c) Sketch the graph of $y = P(t)$ for $0 \le t \le 6$.

d) What is the range of the function? Explain the meaning of the range in terms of a person's blood pressure.

10. A pendulum swings back and forth 10 times in 8 s. It swings through a total horizontal distance of 40 cm.

a) Sketch a graph of this motion for two cycles, beginning with the pendulum at the end of its swing.

b) Describe the transformations necessary to transform $y = \sin x$ into the function you graphed in part a).

c) Write the equation that models this situation.

11. A rung on a hamster wheel, with a radius of 25 cm, is travelling at a constant speed. It makes one complete revolution in 3 s. The axle of the hamster wheel is 27 cm above the ground.

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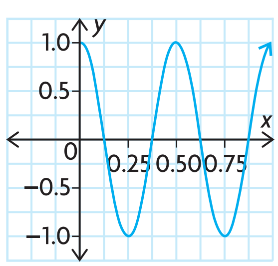
- **a)** Sketch a graph of the height of the rung above the ground during two complete revolutions, beginning when the rung is closest to the ground.
- **b)** Describe the transformations necessary to transform $y = \cos x$ into the function you graphed in part a).
- **c)** Write the equation that models this situation.

12. The graph of a sinusoidal function has been horizontally compressed and horizontally translated to the left. It has maximums at the points $\left(-\frac{5\pi}{7}, 1\right)$ and $\left(-\frac{3\pi}{7}, 1\right)$, and it has a minimum at $\left(-\frac{4\pi}{7}, -1\right)$. If the x-axis is in radians, what is the period of the function?

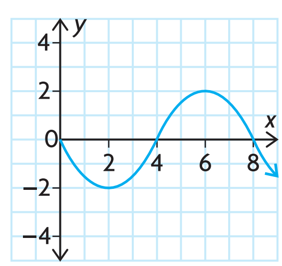
13. The graph of a sinusoidal function has been vertically stretched, vertically translated up, and horizontally translated to the right. The graph has a maximum at $\left(\frac{\pi}{13}, 13\right)$, and the equation of the axis is $y = 9$. If the x-axis is in radians, list one point where the graph has a minimum.

14. Determine a sinusodial equation for each of the following graphs.

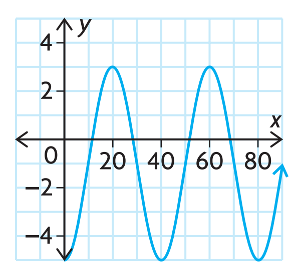
a)



b)



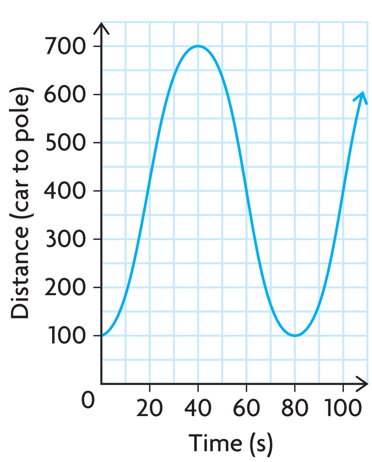
c)



15. Create a flow chart that summarises how you would use transformations to sketch the graph of $f(x) = -2 \sin\left(0.5\left(x - \frac{\pi}{4}\right)\right) + 3$.

C

Extending



16. The graph shows the distance from a light pole to a car racing around a circular track. The track is located north of the light pole.

- **a)** Determine the distance from the light pole to the edge of the track.
- **b)** Determine the distance from the light pole to the centre of the track.
- **c)** Determine the radius of the track.
- **d)** Detemine the time that the car takes to complete one lap of the track.
- **e)** Determine the speed of the car in metres per second.