6.5 Exploring Graphs of the Reciprocal Trigonometric Functions

YOU WILL NEED

- graph paper
- graphing calculator

GOAL

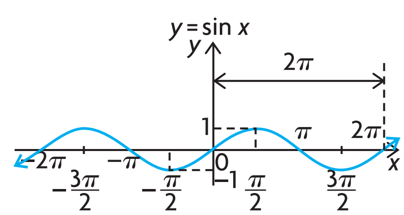
Graph the reciprocal trigonometric functions and determine their key characteristics.

EXPLORE the Math

Recall that the characteristics of the graph of a reciprocal function of a linear or quadratic function are directly related to the characteristics of the original function. Therefore, the key characteristics of the graph of a linear or quadratic function can be used to graph the related reciprocal function. The same strategies can be used to graph the reciprocal of a trigonometric function.

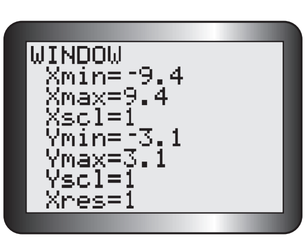
? What do the graphs of the reciprocal trigonometric functions $y = \csc x$, $y = \sec x$, and $y = \cot x$ look like, and what are their key characteristics?

A. Here is the graph of $y = \sin x$.



Use this graph to predict where each of the following characteristics of the graph of $y = \frac{1}{\sin x}$ will occur.

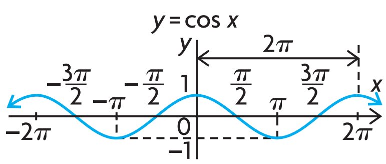
a) vertical asymptotes
b) maximum and minimum values
c) positive and negative intervals
d) intervals of increase and decrease
e) points of intersection for $y = \sin x$ and $y = \frac{1}{\sin x}$



B. Use your predictions in part A to sketch the graph of $y = \frac{1}{\sin x}$ (that is, $y = \csc x$). Verify your sketch by entering $y = \sin x$ into Y1 and $y = \frac{1}{\sin x}$ into Y2 of a graphing calculator, using the window settings shown. Compare the period and amplitude of each function.

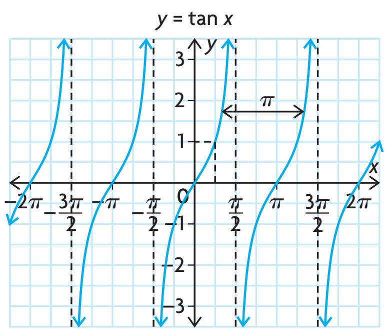
C. Predict what will happen if the period of $y = \sin x$ changes from 2π to π. Change Y1 to $y = \sin\,(2x)$ and Y2 to $y = \frac{1}{\sin\,(2x)}$ and discuss the results.

D. Here is the graph of $y = \cos x$.



Repeat parts A to C using the cosine function and its reciprocal $y = \frac{1}{\cos x}$ (that is, $y = \sec x$).

E. Here is the graph of $y = \tan x$. Recall that $\tan x = \frac{\sin x}{\cos x}$.



Repeat parts A to C using this form of the tangent function and its reciprocal $y = \frac{\cos x}{\sin x}$ (that is, $y = \cot x$).

Reflecting

F. Do the primary trigonometric functions and their reciprocal functions have the same kind of relationship that linear and quadratic functions and their reciprocal functions have? Explain.

G. Which x-values of the reciprocal function, in the interval $-2\pi \leq x \leq 2\pi$, result in vertical asymptotes? Why does this happen?

H. What is the relationship between the positive and negative intervals of the primary trigonometric functions and the positive and negative intervals of their reciprocal functions?

I. Where do the points of intersection occur for the primary trigonometric functions and their reciprocal functions?

In Summary

Key Idea

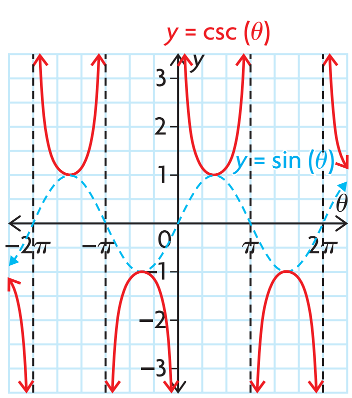
- Each of the primary trigonometric graphs has a corresponding reciprocal function.

Cosecant	Secant	Cotangent
$y = \csc \theta$	$y = \sec \theta$	$y = \cot \theta$
$y = \dfrac{1}{\sin \theta}$	$y = \dfrac{1}{\cos \theta}$	$y = \dfrac{1}{\tan \theta} = \dfrac{\cos \theta}{\sin \theta}$

Need to Know

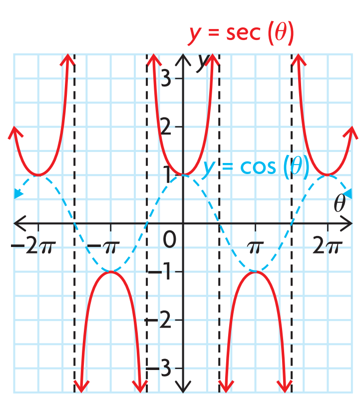
- The graph of a reciprocal trigonometric function is related to the graph of its corresponding primary trigonometric function in the following ways:
  - The graph of the reciprocal function has a vertical asymptote at each zero of the corresponding primary trigonometric function.
  - The reciprocal function has the same positive/negative intervals as the corresponding primary trigonometric function.
  - Intervals of increase on the primary trigonometric function are intervals of decrease on the corresponding reciprocal function. Intervals of decrease on the primary trigonometric function are intervals of increase on the corresponding reciprocal function.
  - The ranges of the primary trigonometric functions include 1 and -1, so a reciprocal function intersects its corresponding primary function at points where the y-coordinate is 1 or -1.
  - If the primary trigonometric function has a local minimum point, the corresponding reciprocal function has a local maximum point at the same θ value. If the primary trigonometric function has a local maximum point, the corresponding reciprocal function has a local minimum point at the same θ value.

Cosecant



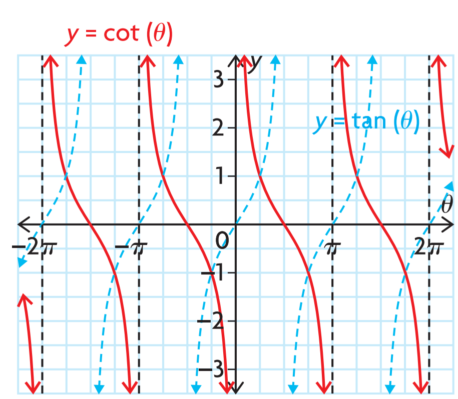
- has vertical asymptotes at the points where $\sin \theta = 0$
- has the same period (2π) as $y = \sin \theta$
- has the domain $\{x \in \mathbf{R} \mid \theta \neq n\pi, n \in \mathbf{I}\}$
- has the range $\{y \in \mathbf{R} \mid |y| \geq 1\}$

Secant



- has vertical asymptotes at the points where $\cos \theta = 0$
- has the same period (2π) as $y = \cos \theta$
- has the domain $\{x \in \mathbf{R} \mid \theta \neq (2n - 1)\frac{\pi}{2}, n \in \mathbf{I}\}$
- has the range $\{y \in \mathbf{R} \mid |y| \geq 1\}$

Cotangent



- has vertical asymptotes at the points where $\tan \theta = 0$
- has zeros at the points where $y = \tan \theta$ has asymptotes
- has the same period (π) as $y = \tan \theta$
- has the domain $\{x \in \mathbf{R} \mid \theta \neq n\pi, n \in \mathbf{I}\}$
- has the range $\{y \in \mathbf{R}\}$

FURTHER Your Understanding

1. The equation $t_n = a + (n - 1)d$ can be used to represent the general term of any arithmetic sequence, where a is the first term and d is the common difference. Use this equation to find an expression that describes the location of each of the following values for $y = \csc x$, where $n \in \mathbf{I}$ and x is in radians.

a) vertical asymptotes
b) maximum values
c) minimum values

2. Find an expression that describes the location of each of the following values for $y = \sec x$, where $n \in \mathbf{I}$ and x is in radians.

a) vertical asymptotes
b) maximum values
c) minimum values

3. Find an expression that describes the location of each of the following values for $y = \cot x$, where $n \in \mathbf{I}$ and x is in radians.

a) vertical asymptotes
b) x-intercepts

4. Use graphing technology to graph $y = \csc x$ and $y = \sec x$. For which values of the independent variable do the graphs intersect? Compare these values with the intersections of $y = \sin x$ and $y = \cos x$. Explain.

5. The graphs of the functions $y = \sin x$ and $y = \cos x$ are congruent, related by a translation of $\frac{\pi}{2}$ where $\sin\left(x + \frac{\pi}{2}\right) = \cos x$. Does this relationship hold for $y = \csc x$ and $y = \sec x$? Verify your conjecture using graphing technology.

6. Two successive transformations can be applied to the graph of $y = \tan x$ to obtain the graph of $y = \cot x$. There is more than one way to apply these transformations, however. Describe one of these compound transformations.

7. Use transformations to sketch the graph of each function. Then state the period of the function.

a) $y = \cot\left(\frac{x}{2}\right)$
b) $y = \csc\left(2\left(x + \frac{\pi}{2}\right)\right)$
c) $y = \sec x - 1$
d) $y = \csc(0.5x + \pi)$