6.6

Modelling with Trigonometric Functions

YOU WILL NEED

- graphing calculator or graphing software

GOAL

Model and solve problems that involve trigonometric functions and radian measurement.

LEARN ABOUT the Math

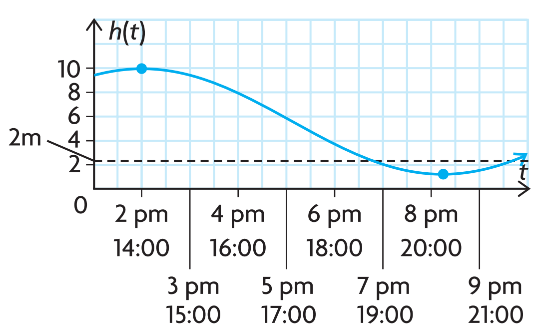
The tides at Cape Capstan, New Brunswick, change the depth of the water in the harbour. On one day in October, the tides have a high point of approximately 10 m at 2 p.m. and a low point of approximately 1.2 m at 8:15 p.m. A particular sailboat has a *draft* of 2 m. This means it can only move in water that is at least 2 m deep. The captain of the sailboat plans to exit the harbour at 6:30 p.m.

? Can the captain exit the harbour safely in the sailboat at 6 p.m.?

EXAMPLE 1 Modelling the problem using a sinusoidal equation

Create a sinusoidal function to model the problem, and use it to determine whether the sailboat can exit the harbour safely at 6 p.m.

Solution



A sinusoidal function can be used to model the height of the water versus time. Draw a sketch to get an idea of when the captain needs to leave. It appears that the captain will have enough depth at 6:30 p.m., but you cannot be sure from a rough sketch.

$$H(t) = a\cos(k(t - d)) + c$$

$$a = \frac{10 - 1.2}{2}$$

$$a = 4.4$$

Choose the cosine function to model the problem, since the graph starts at a maximum value. The amplitude, period, horizontal translation, and equation of the axis need to be determined.

Use the maximum and minimum measurements of the tides to calculate the amplitude of the function. This gives the value of *a* in the equation.

$$\text{Period} = \frac{2\pi}{k}$$
$$12.5 = \frac{2\pi}{k}$$
$$12.5k = 2\pi$$
$$k = \frac{2\pi}{12.5} = \frac{4\pi}{25}$$

In a sinusoidal function, the horizontal distance between the maximum and minimum points represents half of one cycle.

Since a maximum tide and a minimum tide occur 6 h 15 min apart, the period must be 12.5 h. The period can be used to determine the value of k in the equation.

$$c = \frac{10 + 1.2}{2}$$
$$c = 5.6$$

The equation of the axis is the mean of the maximum and minimum points. This can be used to determine the value of c in the equation.

A function that models the tides at Cape Capstan is

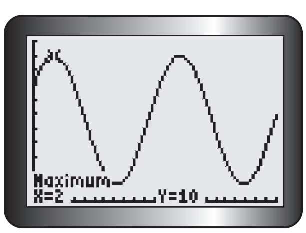
$$H(t) = 4.4\cos\left(\frac{4\pi}{25}(t - 2)\right) + 5.6.$$

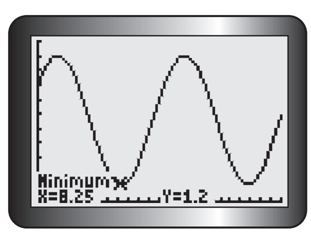
The parent cosine function starts at a maximum point.
If we let $t = 0$ represent noon, then our function needs a maximum at $t = 2$ (or 2 p.m.).
We use a horizontal translation right 2 units.
Therefore $d = 2$.

$$H(18) = 4.4\cos\left(\frac{4\pi}{25}(6.5 - 2)\right) + 5.6$$
$$= 4.4\cos\left(\frac{18\pi}{25}\right) + 5.6$$
$$\doteq 2.80 \text{ m}$$

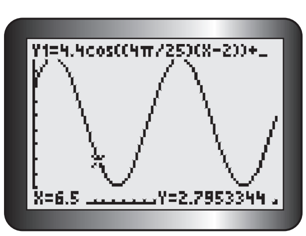
To determine the water level at 6:30 p.m., let $t = 6.5$.

Since the depth of the water is greater than 2 m at 6:30 p.m., the sailboat can safely exit the harbour.





To verify the solution, enter the function in the equation editor as Y1.
Use the value operation to confirm a high tide at 2 p.m. $(t = 2)$ and a low tide at 8:15 p.m. $(t = 8.25)$.



Use the value operation to determine the depth of the water at 6:30 p.m. $(t = 6.5)$.

Reflecting

A. What characteristics of your model would change if you used a sine function to model the problem?

B. What role did the maximum value play in determining the required horizontal translation?

C. If $t = 0$ was set at 2 p.m. instead of noon, how would the equation change? Would this make a difference to your final answer?

APPLY the Math

EXAMPLE 2 Representing a situation described by data using a sinusoidal equation

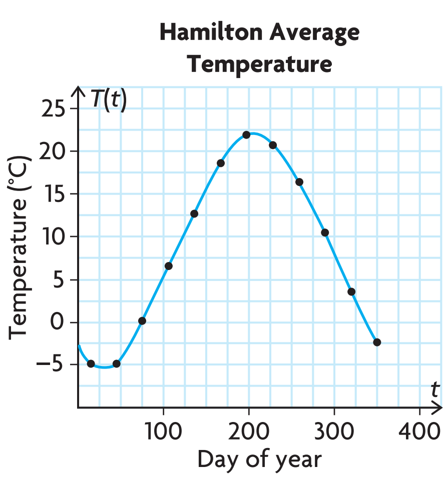
The following table shows the average monthly means of the daily (24 h) temperatures in Hamilton, Ontario. Each month's average temperature is represented by the day in the middle of the month.

Month	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sep.	Oct.	Nov.	Dec.
Day of Year	15	45	75	106	136	167	197	228	259	289	320	350
°C	−4.8	−4.8	−0.2	6.6	12.7	18.6	21.9	20.7	16.4	10.5	3.6	−2.3

a) Plot the temperature data for Hamilton, and fit a sinusoidal curve to the points.
b) Estimate the average daily temperature in Hamilton on the 200th day of the year.

Solution A: Using the data and reasoning about the characteristics of the graph

a)



Plot the data, and sketch a smooth curve through the points.

The curve appears to be sinusoidal, so use $y = a \sin(k(t - d)) + c$ as the model for this situation.

$$a = \frac{\text{maximum} - \text{minimum}}{2}$$

$$a = \frac{21.9 - (-4.8)}{2}$$

$$a = 13.35$$

Estimate the maximum and minimum temperatures for the year from the graph. Use these temperatures to calculate the values of *a* and *c*. The value of *a* gives the amplitude. The sine function has been stretched vertically by a factor of 13.35.

$$c = \frac{\text{maximum} + \text{minimum}}{2}$$

$$c = \frac{21.9 + (-4.8)}{2}$$

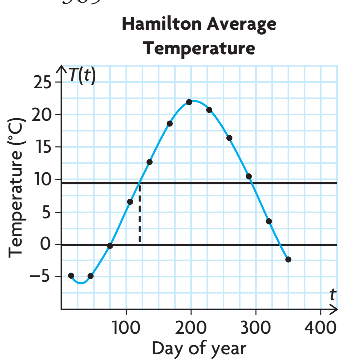
$$c = 8.55$$

The value of *c* gives the horizontal axis. The sine function has been vertically translated by 8.55 units on the Temperature axis. Lightly draw a horizontal line through your graph at this value.

$$\text{Period} = \frac{2\pi}{k}, \text{ so } k = \frac{2\pi}{\text{period}}$$

$$k = \frac{2\pi}{365}$$

The value of *k* in the equation is determined by the period. Assume that the cycle repeats itself every year (365 days).

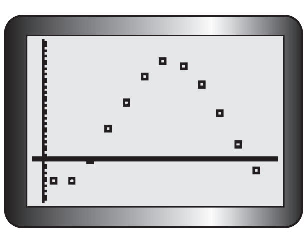


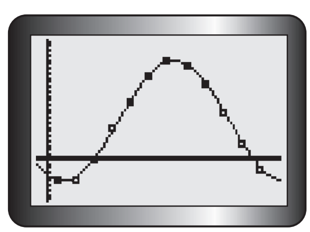
To determine the value of *d*, estimate where the horizontal axis first intersects the curve.

Since this graph appears to have been translated to the right, $d \doteq 116$.

$$T(t) = 13.35 \sin\left(\frac{2\pi}{365}(t - 116)\right) + 8.55$$

Replace the parameters in the general sine equation.





Verify the result by entering the data into L1 and L2 in a graphing calculator and creating a scatter plot. Enter the sine function into Y1 and observe that it matches the data.

b) $$T(t) = 13.35 \sin\left(\frac{2\pi}{365}(t - 116)\right) + 8.55$$

$$T(200) = 13.35 \sin\left(\frac{2\pi}{365}(200 - 116)\right) + 8.55$$

$$\doteq 21.8\,°\text{C}$$

Let $t = 200$, and evaluate the sine function.

This model predicts that the average daily temperature in Hamilton on the 200th day of the year is about 21.8 °C.

Since sinusoidal functions are periodic, they can be used (where appropriate) to make educated predictions.

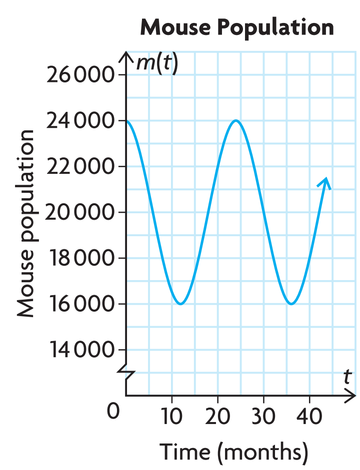
EXAMPLE 3 Analyzing a situation that involves sinusoidal models

The population size, O, of owls (predators) in a certain region can be modelled by the function $O(t) = 1000 + 100 \sin\left(\frac{\pi t}{12}\right)$, where t represents the time in months and $t = 0$ represents January. The population size, m, of mice (prey) in the same region is given by the function $m(t) = 20\,000 + 4000 \cos\left(\frac{\pi t}{12}\right)$.

a) Sketch the graphs of these functions.
b) Compare the graphs, and discuss the relationships between the two populations.
c) How does the mice-to-owls ratio change over time?
d) When is there the most food per owl? When is it safest for the mice?

Solution

a) Graph the prey function.



$m(t) = 4000 \cos\left(\frac{\pi t}{12}\right) + 20\,000.$

The mouse population has a maximum of 24 000 and a minimum of 16 000.

$a = 4000$

The amplitude of the curve is 4000.

$c = 20\,000$

The axis is the line $m(t) = 20\,000$.

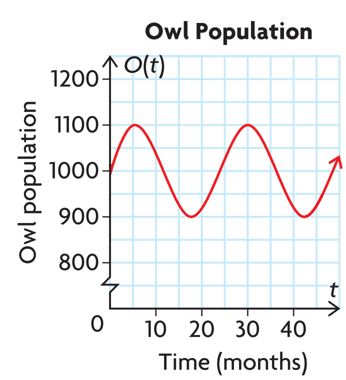
$k = \frac{\pi}{12}$, so the period $= \frac{2\pi}{k}$

$$\text{period} = \frac{2\pi}{\frac{\pi}{12}}$$

$$\text{period} = 2\pi \times \frac{12}{\pi} = 24$$

The period is 24 months.

Graph the predator function.



$O(t) = 100 \sin\left(\frac{\pi t}{12}\right) + 1000.$

The owl population has a maximum of 1100 and a minimum of 900.

$a = 100$

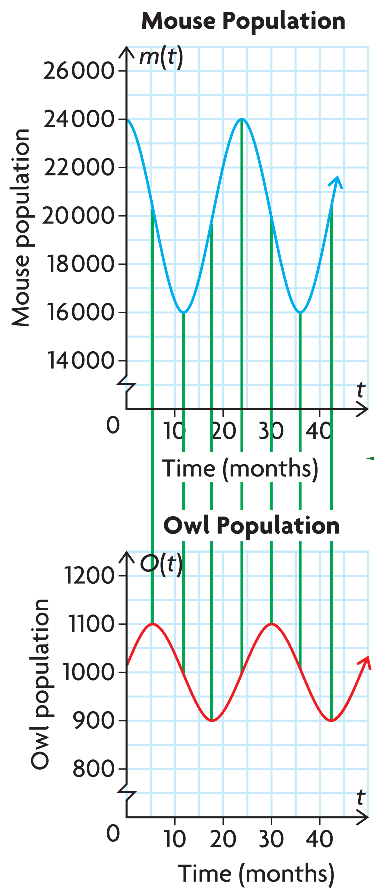
The amplitude of the curve is 100.

$c = 1000$

The axis is the line $O(t) = 1000$.

$k = \frac{\pi}{12}$ as above, so this period is also 24 months.

b)

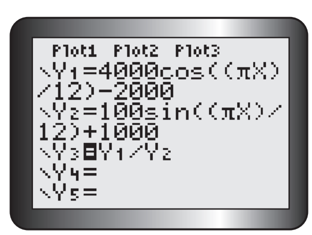


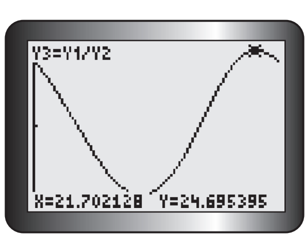
The graphs can be compared, since the same scale was used on both horizontal axes. As the owl population begins to increase, the mouse population begins to decrease. The mouse population continues to decrease, and this has an impact on the owl population, since its food supply dwindles. The owl population peaks and then also starts to decrease. The mouse population reaches a minimum and begins to rise as there are fewer owls to eat the mice. As the mouse population increases, food becomes more plentiful for the owls. So their population begins to rise again. Since both graphs have the same period, this pattern repeats every 24 months.

c) The following table shows the ratio of mice to owls at key points in the first four years.

Time	Mice	Owls	Mice-to-Owl Ratio
0	24 000	1000	24
6	20 000	1100	18.2
12	16 000	1000	16
18	20 000	900	22.2
24	24 000	1000	24

There seems to be a pattern. Enter the mouse function into Y1 of the equation editor of a graphing calculator, and enter the owl function into Y2. Turn off each function, and enter Y3 = Y1/Y2.





The resulting graph is shown. The ratio of mice to owls is also sinusoidal.

d) The most food per owl occurs when the ratio of mice to owls is the highest (there are more mice per owl).

The safest time for the mice occurs at the same time, when the ratio of mice to owls is the highest (there are fewer owls per mouse).

This occurs near the end of the 21st month of the two-year cycle.

In Summary

Key Ideas

- The graphs of $y = \sin x$ and $y = \cos x$ can model periodic phenomena when they are transformed to fit a given situation. The transformed functions are of the form $y = a \sin (k(x - d)) + c$ and $y = a \cos (k(x - d)) + c$, where
  - $|a|$ is the amplitude and $a = \frac{\text{max} - \text{min}}{2}$
  - $|k|$ is the number of cycles in 2π radians, when the period $= \frac{2\pi}{k}$
  - d gives the horizontal translation
  - c is the vertical translation and $y = c$ is the horizontal axis

Need to Know

- Tables of values, graphs, and equations of sinusoidal functions can be used as mathematical models when solving problems. Determining the equation of the appropriate sine or cosine function from the data or graph provided is the most efficient strategy, however, since accurate calculations can be made using the equation.

CHECK Your Understanding

1. A cosine curve has an amplitude of 3 units and a period of 3π radians. The equation of the axis is $y = 2$, and a horizontal shift of $\frac{\pi}{4}$ radians to the left has been applied. Write the equation of this function.

2. Determine the value of the function in question 1 if $x = \frac{\pi}{2}, \frac{3\pi}{4}$, and $\frac{11\pi}{6}$.

3. Sketch a graph of the function in question 1. Use your graph to estimate the x-value(s) in the domain $0 < x < 2$, where $y = 2.5$, to one decimal place.

PRACTISING

4. The height of a patch on a bicycle tire above the ground, as a function of time, is modelled by one sinusoidal function. The height of the patch above the ground, as a function of the total distance it has travelled, is modelled by another sinusoidal function. Which of the following characteristics do the two sinusoidal functions share: amplitude, period, equation of the axis?

5. Mike is waving a sparkler in a circular motion at a constant speed. **K** The tip of the sparkler is moving in a plane that is perpendicular to the ground. The height of the tip of the sparkler above the ground, as a function of time, can be modelled by a sinusoidal function. At $t = 0$, the sparkler is at its highest point above the ground.

- **a)** What does the amplitude of the sinusoidal function represent in this situation?
- **b)** What does the period of the sinusoidal function represent in this situation?
- **c)** What does the equation of the axis of the sinusoidal function represent in this situation?
- **d)** If no horizontal translations are required to model this situation, should a sine or cosine function be used?

6. To test the resistance of a new product to temperature changes, the **A** product is placed in a controlled environment. The temperature in this environment, as a function of time, can be described by a sine function. The maximum temperature is 120 °C, the minimum temperature is −60 °C, and the temperature at $t = 0$ is 30 °C. It takes 12 h for the temperature to change from the maximum to the minimum. If the temperature is initially increasing, what is the equation of the sine function that describes the temperature in this environment?

7. A person who was listening to a siren reported that the frequency of the sound fluctuated with time, measured in seconds. The minimum frequency that the person heard was 500 Hz, and the maximum frequency was 1000 Hz. The maximum frequency occurred at $t = 0$ and $t = 15$. The person also reported that, in 15, she heard the maximum frequency 6 times (including the times at $t = 0$ and $t = 15$). What is the equation of the cosine function that describes the frequency of this siren?

8. A contestant on a game show spins a wheel that is located on a plane perpendicular to the floor. He grabs the only red peg on the circumference of the wheel, which is 1.5 m above the floor, and pushes it downward. The red peg reaches a minimum height of 0.25 m above the floor and a maximum height of 2.75 m above the floor. Sketch two cycles of the graph that represents the height of the red peg above the floor, as a function of the total distance it moved. Then determine the equation of the sine function that describes the graph.

9. At one time, Maple Leaf Village (which no longer exists) had North America's largest Ferris wheel. The Ferris wheel had a diameter of 56 m, and one revolution took 2.5 min to complete. Riders could see Niagara Falls if they were higher than 50 m above the ground. Sketch three cycles of a graph that represents the height of a rider above the ground, as a function of time, if the rider gets on at a height of 0.5 m at $t = 0$ min. Then determine the time intervals when the rider could see Niagara Falls.

10. The number of hours of daylight in Vancouver can be modelled by a sinusoidal function of time, in days. The longest day of the year is June 21, with 15.7 h of daylight. The shortest day of the year is December 21, with 8.3 h of daylight.
    a) Find an equation for $n(t)$, the number of hours of daylight on the nth day of the year.
    b) Use your equation to predict the number of hours of daylight in Vancouver on January 30th.

11. **T** The city of Thunder Bay, Ontario, has average monthly temperatures that vary between -14.8 °C and 17.6 °C. The following table gives the average monthly temperatures, averaged over many years. Determine the equation of the sine function that describes the data, and use your equation to determine the times that the temperature is below 0 °C.

Month	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sep.	Oct.	Nov.	Dec.
Average Temperature (°C)	−14.8	−12.7	−5.9	2.5	8.7	13.9	17.6	16.5	11.2	5.6	−2.7	−11.1

12. **C** A nail is stuck in the tire of a car. If a student wanted to graph a sine function to model the height of the nail above the ground during a trip from Kingston, Ontario, to Hamilton, Ontario, should the student graph the distance of the nail above the ground as a function of time or as a function of the total distance travelled by the nail? Explain your reasoning.

Extending

13. A clock is hanging on a wall, with the centre of the clock 3 m above the floor. Both the minute hand and the second hand are 15 cm long. The hour hand is 8 cm long. For each hand, determine the equation of the cosine function that describes the distance of the tip of the hand above the floor as a function of time. Assume that the time, t, is in minutes and that the distance, $D(t)$, is in centimetres. Also assume that $t = 0$ is midnight.