6.7 Rates of Change in Trigonometric Functions

GOAL

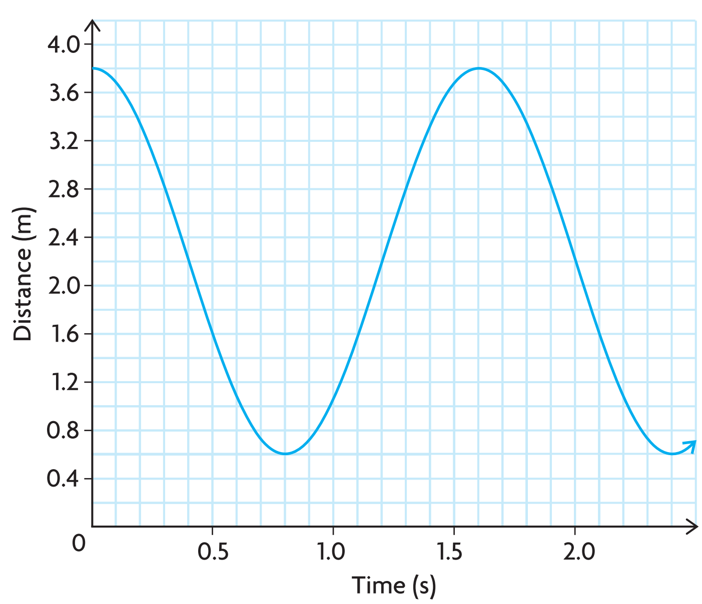
Examine average and instantaneous rates of change in trigonometric functions.

LEARN ABOUT the Math

Melissa used a motion detector to measure the horizontal distance between her and a child on a swing. She stood in front of the child and recorded the distance, $d(t)$, in metres over a period of time, t, in seconds. The data she collected are given in the following tables and are shown on the graph below.

Time (s)	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1	1.1
Distance (m)	3.8	3.68	3.33	2.81	2.2	1.59	1.07	0.72	0.6	0.72	1.07	1.59

Time (s)	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0	2.1	2.2	2.3	2.4
Distance (m)	2.2	2.81	3.33	3.68	3.8	3.68	3.33	2.81	2.2	1.59	1.07	0.72	0.6



? How did the speed of the child change as the child swung back and forth?

EXAMPLE 1 Using the data and the graph to analyze the situation

Use the data and the graph to discuss how the speed of the child changed as the child swung back and forth.

Solution

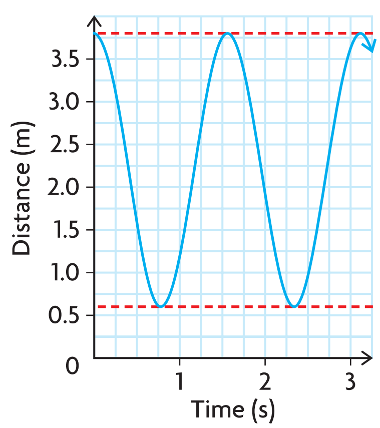
Analyze the motion.

Melissa began recording the motion when the child was the farthest distance from the motion detector, which was 3.8 m. The child's closest distance to the motion detector was 0.6 m and occurred at 0.8 s. The child was moving toward the motion detector between 0 s and 0.8 s and away from the motion detector between 0.8 s and 1.6 s.

Looking at the graph, the maximum value was 3.8 and occurred at 0 s and 1.6 s. It took 1.6 s for the child to swing one complete cycle.

Looking at the data and the graph, the distances between the child and the motion detector were decreasing between 0 s and 0.8 s, and increasing between 0.8 s and 1.6 s. This pattern repeated itself every multiple of 1.6 s.

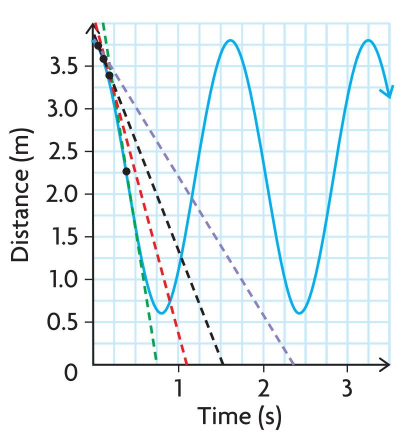
Analyze the instantaneous velocity by drawing tangent lines at various points over one swing cycle.



The slope of a tangent line on any distance versus time graph gives the instantaneous velocity, which is the instantaneous rate of change in distance with respect to time.

When the child was at the farthest point and closest point from the motion detector, the instantaneous velocity was 0.

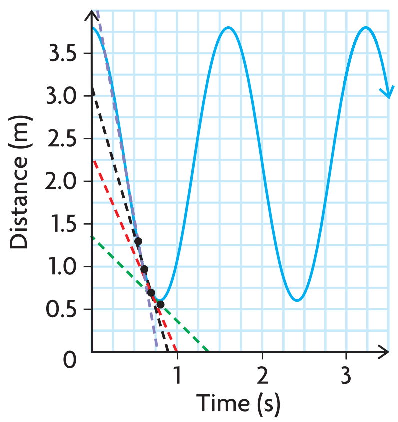
Between 0 s and about 0.4 s, the child's speed was increasing.



On this interval, the tangent lines become steeper as time increases.

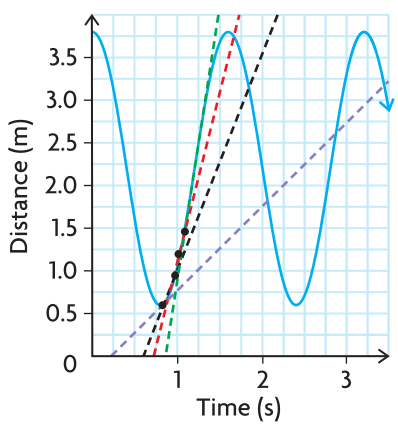
Speed $= |\text{velocity}| = \left|\dfrac{\Delta\text{distance}}{\Delta\text{time}}\right|$. This means the magnitudes of the slopes are increasing. The tangent lines have negative slopes, which means the distance between the child and the motion detector continues to decrease.

Between 0.4 s and about 0.8 s, the child's speed was decreasing.



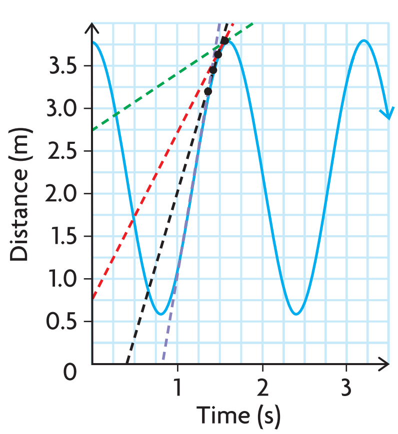
On this interval, the tangent lines are getting less steep as time increases. This means the magnitudes of the slopes are decreasing. The tangent lines still have negative slopes, which means the distance between the child and the motion detector is still decreasing. The child is slowing down as the swing approaches the point where a change in direction occurs. The slopes indicate a change in the child's position from toward the detector to away from the detector.

Between 0.8 s and about 1.2 s, the child's speed was increasing.



On this interval, the tangent lines are getting steeper as time increases. This means the magnitudes of the slopes are increasing. The tangent lines have positive slopes, which means the distance between the child and the motion detector is increasing. Therefore, the motion is away from the detector.

Between 1.2 s and about 1.6 s, the child's speed was decreasing.



On this interval, the tangent lines are getting less steep as time increases. This means the magnitudes of the slopes are decreasing. The tangent lines still have positive slopes, which means the distance between the child and the motion detector is still increasing. The child is slowing down as the swing approaches the point where there is a change in direction from away from the detector to toward the detector.

Reflecting

A. Explain how the data in the table indicates the direction in which the child swung.

B. Explain how the sign of the slope of each tangent line indicates the direction in which the child swung.

C. How can you tell, from the graph, when the speed of the child was 0 m/s?

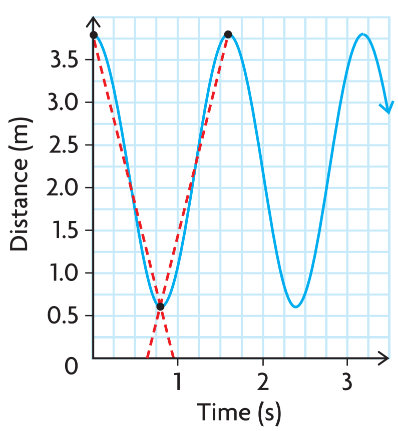
D. If someone began to push the child after 2.4 s, describe what effect this would have on the distance versus time graph.

APPLY the Math

EXAMPLE 2 Using the slopes of secant lines to calculate average rate of change

Calculate the child's average speed over the intervals of time as the child swung toward and away from the motion detector on the first swing.

Solution



The absolute value of the slope of a secant line on any distance versus time graph gives the average rate of change in distance, with respect to time or average speed.

The secant line that is decreasing has a negative slope, indicating that the distance between the child and the motion detector was decreasing between 0 s and 0.8 s.

The secant line that is increasing has a positive slope, indicating that the distance between the child and the motion detector was increasing between 0.8 s and 1.6 s.

Interval	$\frac{\Delta \text{ Distance}}{\Delta \text{ Time}}$	Average Speed (m/s)
$0 \le t \le 0.8$	$\left\|\frac{0.6 - 3.8}{0.8 - 0}\right\|$	$\|-4\| = 4$
$0.8 \le t \le 1.6$	$\left\|\frac{3.8 - 0.6}{1.6 - 0.8}\right\|$	$\|4\| = 4$

Use the data in the table and the relationship $\left|\frac{\Delta \text{ distance}}{\Delta \text{ time}}\right|$ to calculate the average speed.

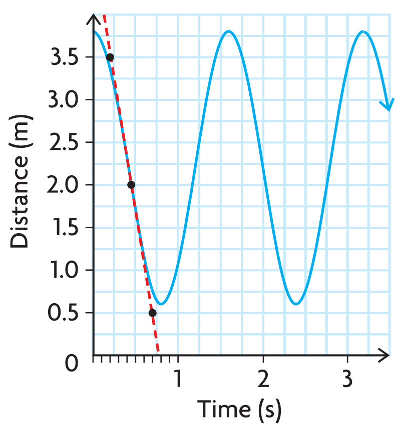
The child's average speed was the same in both directions as the child swung back and forth.

EXAMPLE 3 Using the difference quotient to estimate instantaneous rates of change

To model the motion of the child on the swing, Melissa determined that she could use the equation $d(t) = 1.6\cos\left(\frac{\pi}{0.8}t\right) + 2.2$, where $d(t)$ is the distance from the child to the motion detector, in metres, and t is the time, in seconds. Use this equation to estimate when the child was moving the fastest and what speed the child was moving at this time.

Solution

The child must have been moving the fastest at around 0.4 s. Drawing a tangent line at $t = 0.4$ supports this, since the tangent line appears to be steepest here.



The child moved slowest near the ends of the swing, approaching the points where a change in direction occurred. At these points, the tangent lines are horizontal so their slopes are 0. The child's speed was 0 m/s at 0 s, 0.8 s, and 1.6 s. The child's speed increased between 0 s and 0.4 s, and then decreased between 0.4 s and 0.8 s.

Estimate the coordinates of two points on the tangent line to estimate its slope. Use (0.2, 3.5) and (0.7, 0.5).

$$\text{Slope} = \frac{0.5 - 3.5}{0.7 - 0.2} = -6$$

The child was moving at about 6 m/s.

To get a better estimate of the child's speed at this time, use the difference quotient $\frac{d(a + h) - d(a)}{h}$, where $a = 0.4$. Use a very small value for h.

$$\text{Speed} = \left|\frac{d(0.4 + h) - d(0.4)}{h}\right|$$

Let $h = 0.001$.

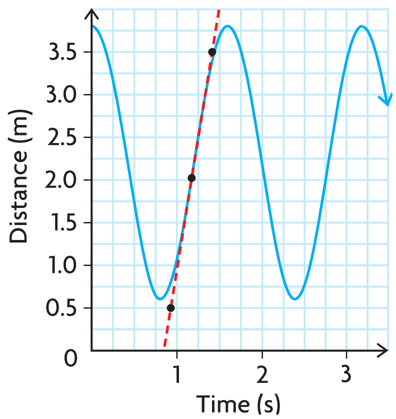
$$\begin{aligned}\text{Speed} &= \left|\frac{d(0.4 + 0.001) - d(0.4)}{0.001}\right| \\ &= \left|\frac{d(0.401) - d(0.4)}{0.001}\right| \\ &= \left|\frac{\left[1.6\cos\left(\frac{\pi}{0.8}(0.401)\right) + 2.2\right] - \left[1.6\cos\left(\frac{\pi}{0.8}(0.4)\right) + 2.2\right]}{0.001}\right| \\ &\doteq \left|\frac{2.19372 - 2.2}{0.001}\right| \\ &\doteq |-6.28| \text{ or } 6.28\end{aligned}$$

This speed is about 23 km/h.

The child's fastest speed was about 6.3 m/s.

The child was also travelling the fastest at around 1.2 s. Drawing a tangent line at $t = 1.2$ supports this, since the tangent line appears to be steepest here.

The child's speed increased between 0.8 s and 1.2 s, and then decreased between 1.2 s and 1.6 s.



Estimate the coordinates of two points on the tangent line to estimate the slope of the line.

Use $(0.9, 0.5)$ and $(1.4, 3.5)$.

$$\text{Slope} = \frac{3.5 - 0.5}{1.4 - 0.9} = 6$$

The child was moving at about 6 m/s.

$$\text{Speed} = \left|\frac{d(1.2 + h) - d(1.2)}{h}\right|$$

Let $h = 0.001$.

$$\text{Speed} = \left|\frac{d(1.2 + 0.001) - d(1.2)}{0.001}\right|$$

$$= \left|\frac{d(1.201) - d(1.2)}{0.001}\right|$$

$$= \left|\frac{\left[1.6 \cos\left(\frac{\pi}{0.8}(1.201)\right) + 2.2\right] - \left[1.6 \cos\left(\frac{\pi}{0.8}(1.2)\right) + 2.2\right]}{0.001}\right|$$

$$= \left|\frac{2.20628 - 2.2}{0.001}\right|$$

$$\doteq |6.28| \text{ or } 6.28$$

The child's fastest speed was about 6.3 m/s.

To improve the estimate of the child's speed at this time, use the difference quotient $\frac{d(a + h) - d(a)}{h}$, where $a = 1.2$. Use a very small value for h.

In Summary

Key Idea

- The average and instantaneous rates of change of a sinusoidal function can be determined using the same strategies that were used for other types of functions.

Need to Know

- The tangent lines at the maximum and minimum values of a sinusoidal function are horizontal. Since the slope of a horizontal line is zero, the instantaneous rate of change at these points is zero.
- In a sinusoidal function, the slope of a tangent line is the least at the point that lies halfway between the maximum and minimum values. The slope is the greatest at the point that lies halfway between the minimum and maximum values. As a result, the instantaneous rate of change at these points is the least and greatest, respectively. The approximate value of the instantaneous rate of change can be determined using one of the strategies below:
  - sketching an approximate tangent line on the graph and estimating its slope using two points that lie on the secant line
  - using two points in the table of values (preferably two points that lie on either side and/or as close as possible to the tangent point) to calculate the slope of the corresponding secant line
  - using the defining equation of the trigonometric function and a very small interval near the point of tangency to calculate the slope of the corresponding secant line

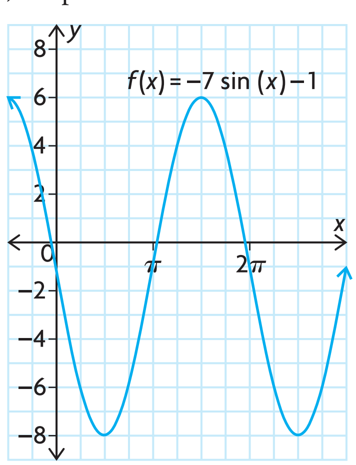
CHECK Your Understanding

1. For the following graph of a function, state two intervals in which the function has an average rate of change in $f(x)$ that is

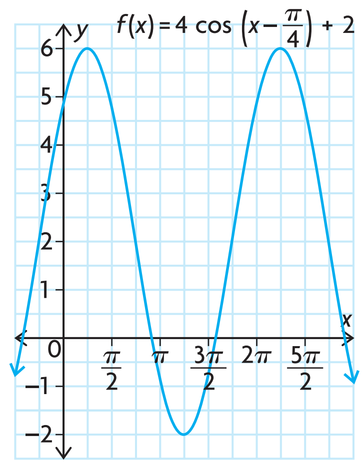
a) zero

b) a negative value

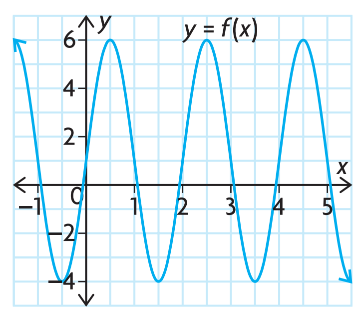
c) a positive value



2. For this graph of a function, state two points where the function has an instantaneous rate of change in $f(x)$ that is
   a) zero
   b) a negative value
   c) a positive value



3. Use the graph to calculate the average rate of change in $f(x)$ on the interval $2 \leq x \leq 5$.



4. Determine the average rate of change of the function $y = 2\cos\left(x - \frac{\pi}{3}\right) + 1$ for each interval.
   a) $0 \leq x \leq \frac{\pi}{2}$
   b) $\frac{\pi}{6} \leq x \leq \frac{\pi}{2}$
   c) $\frac{\pi}{3} \leq x \leq \frac{\pi}{2}$
   d) $\frac{\pi}{2} \leq x \leq \frac{5\pi}{4}$

PRACTISING

5. **K** State two intervals where the function $y = 3\cos(4x) - 4$ has an average rate of change that is
   a) zero
   b) a negative value
   c) a positive value

6. State two points where the function $y = -2 \sin(2\pi x) + 7$ has an instantaneous rate of change that is

a) zero

b) a negative value

c) a positive value

7. State the average rate of change of each of the following functions over the interval $\frac{\pi}{4} \leq x \leq \pi$.

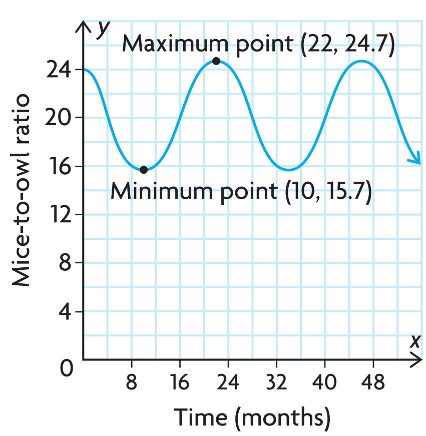
a) $y = 6 \cos(3x) + 2$

b) $y = -5 \sin\left(\frac{1}{2}x\right) - 9$

c) $y = \frac{1}{4} \cos(8x) + 6$

8. T The height of the tip of an airplane propeller above the ground once the airplane reaches full speed can be modelled by a sine function. At full speed, the propeller makes 200 revolutions per second. At $t = 0$, the tip of the propeller is at its minimum height above the ground. Determine whether the instantaneous rate of change in height at $t = \frac{1}{300}$ is a negative value, a positive value, or zero.

9. Recall in Section 6.6, Example 3, the situation that modelled the populations of mice and owls in a particular area.



a) Determine an equation for the curve that models the ratio of mice per owl.

b) Use the curve to determine when the ratio of mice per owl has its fastest and slowest instantaneous rates of change.

c) Use the equation you determined in part a) to estimate the instantaneous rate of change in mice per owl when this rate is at its maximum. Use a centred interval of 1 month before to 1 month after the time when the instantaneous rate of change is at its maximum to make your estimate.

10. The number of tons of paper waiting to be recycled at a 24 h recycling plant can be modelled by the equation $P(t) = 0.5 \sin\left(\frac{\pi}{6}t\right) + 4$, where t is the time, in hours, and $P(t)$ is the number of tons waiting to be recycled.

a) Use the equation to estimate the instantaneous rate of change in tons of paper waiting to be recycled when this rate is at its maximum. To make your estimate, use each of the following centred intervals:

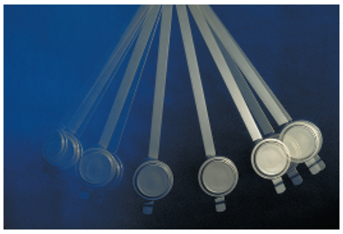
i) 1 h before to 1 h after the time when the instantaneous rate of change is at its maximum

ii) 0.5 h before to 0.5 h after the time when the instantaneous rate of change is at its maximum

iii) 0.25 h before to 0.25 h after the time when the instantaneous rate of change is at its maximum

b) Which estimate is the most accurate? What is the relationship between the size of the interval and the accuracy of the estimate?

11. A strobe photography camera takes photos at regular intervals to capture the motion of a pendulum as it swings from right to left. A student takes measurements from the photo below to analyze the motion.



Time (s)	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
Horizontal Distance from Rest Position* (cm)	7.2	6.85	5.8	4.25	2.2	0.0	−2.2	−4.25	−5.8	−6.85	−7.2

*negative is left of rest position

a) Plot the data, and draw a smooth curve through the points.

b) What portion of one cycle is represented by the curve?

c) Select the endpoints, and determine the average rate of change in horizontal distance on this interval of time.

d) Can you tell, from the photo, when the pendulum bob is moving the fastest? Explain.

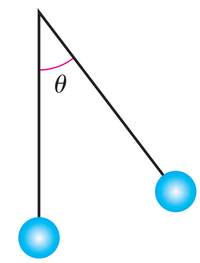
e) Explain how your answer to part d) relates to the rate of change as it is represented on the graph.

12. A ship that is docked in a harbour rises and falls with the waves. The function $h(t) = \sin\left(\frac{\pi}{5}t\right)$ models the vertical movement of the ship, h in metres, at t seconds.
A

a) Determine the average rate of change in the height of the ship over the first 5 s.

b) Estimate the instantaneous rate of change in the height of the ship at $t = 6$.

13. For a certain pendulum, the angle θ shown is given by the equation $\theta = \frac{1}{5}\sin\left(\frac{1}{2}\pi t\right)$ where t is in seconds and θ is in radians.



a) Sketch a graph of the function given by the equation.

b) Calculate the average rate of change in the angle the pendulum swings through in the interval $t \in [0, 1]$.

c) Estimate the instantaneous rate of change in the angle the pendulum swings through at $t = 1.5$ s.

d) On the interval $t \in [0, 8]$, estimate the times when the pendulum's speed is greatest.

14. Compare the instantaneous rates of change of $f(x) = \sin x$ and $f(x) = 3 \sin x$ for the same values of x. What can you conclude? Are there values of x for which the instantaneous rates of change of the two functions are the same?
C

Extending

15. In calculus, the derivative of a function is a function that yields the instantaneous rate of change of a function at any given point.

a) Estimate the instantaneous rate of change of the function $f(x) = \sin x$ for the following values of x: $-\pi$, $-\frac{\pi}{2}$, 0, $\frac{\pi}{2}$, and π.

b) Plot the points that represent the instantaneous rate of change, and draw a sinusoidal curve through them. What function have you graphed? Based on this information, what is the derivative of $f(x) = \sin x$?

16. **a)** Estimate the instantaneous rate of change of the function $f(x) = \cos x$ for the following values of x: $-\pi$, $-\frac{\pi}{2}$, 0, $\frac{\pi}{2}$, and π.

b) Plot the points that represent the instantaneous rate of change, and draw a sinusoidal curve through them. What function have you graphed? Based on this information, what is the derivative of $f(x) = \cos x$?