7.1 Exploring Equivalent Trigonometric Functions

YOU WILL NEED

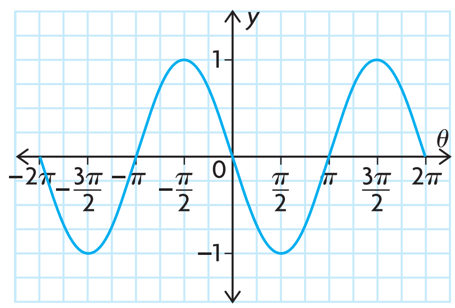
- graphing calculator or graphing software

GOAL

Identify equivalent trigonometric relationships.

EXPLORE the Math

What is a possible equation for the function shown?



Craig, Erin, Robin, and Sarah are comparing their answers to the question shown above.

Craig's function: $f(\theta) = -\sin \theta$

Erin's function: $g(\theta) = \sin (\theta + \pi)$

Robin's function: $h(\theta) = \sin (\theta - \pi)$

Sarah's function: $j(\theta) = \cos \left(\theta + \frac{\pi}{2}\right)$

Their teacher explains that they are all correct because they have written equivalent trigonometric functions.

? How can you verify that these equations are equivalent and identify other equivalent trigonometric expressions?

A. Enter each student's function into Y1 to Y4 in the equation editor of a graphing calculator, using the settings shown. Use radian mode, and graph using the Zoom 7:Ztrig command. What do you notice?

B. Examine the table of values for each function. Are you convinced that the four functions are equivalent? Explain.

Tech *Support*

Scroll to the left of Y2, Y3, and Y4. Press Enter until the required graphing option appears.

Creating equivalent expressions using the period of a function

C. Clear all functions from the calculator, and graph $f(\theta) = \sin \theta$. Using transformations, explain why $\sin (\theta + 2\pi) = \sin \theta$. Write a similar statement for $\cos \theta$ and another similar statement for $\tan \theta$.

D. Verify that your statements for part C are equivalent by graphing the corresponding pair of functions. Write similar statements for the reciprocal trigonometric functions, and verify them by graphing.

Creating equivalent expressions by classifying a function as odd or even

E. $f(\theta) = \cos\theta$ is an **even function** because its graph is symmetrical in the y-axis. Use transformations to explain why $\cos(-\theta) = \cos\theta$, and then verify by graphing.

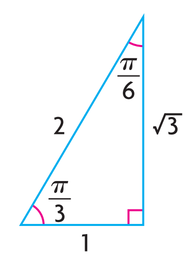
F. $f(\theta) = \sin\theta$ is an **odd function** because its graph has rotational symmetry about the origin. Use transformations to explain why $\sin(-\theta) = -\sin\theta$, and then verify by graphing.

G. Classify the tangent functions as even or odd. Based on your classification, write the corresponding pair of equivalent expressions.

Creating equivalent expressions using complementary angles

H. Determine the exact values of the six trigonometric ratios for each acute angle in the triangle shown. Record the values in a table like the one below. Describe any relationships that you see.

$\sin\left(\frac{\pi}{3}\right) =$	$\csc\left(\frac{\pi}{3}\right) =$	$\sin\left(\frac{\pi}{6}\right) =$	$\csc\left(\frac{\pi}{6}\right) =$
$\cos\left(\frac{\pi}{3}\right) =$	$\sec\left(\frac{\pi}{3}\right) =$	$\cos\left(\frac{\pi}{6}\right) =$	$\sec\left(\frac{\pi}{6}\right) =$
$\tan\left(\frac{\pi}{3}\right) =$	$\cot\left(\frac{\pi}{3}\right) =$	$\tan\left(\frac{\pi}{6}\right) =$	$\cot\left(\frac{\pi}{6}\right) =$



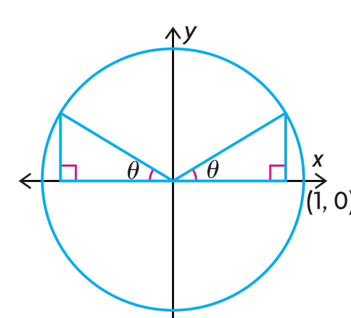
I. Repeat part H for a right triangle in which one acute angle is $\frac{\pi}{8}$ and the other acute angle is $\frac{3\pi}{8}$. Use a calculator to determine the approximate values of the six trigonometric ratios for each of these acute angles. Record the values in a table like the one above. How do the relationships in this table compare with the relationships in the table you completed for part H?

J. Any right triangle, where θ is the measure of one of the acute angles, has a complementary angle of $\left(\frac{\pi}{2} - \theta\right)$ for the other angle. Explain how you know that the cofunction **identity** $\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$ is true.

K. Write all the other cofunction identities between θ and $\left(\frac{\pi}{2} - \theta\right)$ based on the relationships in parts H and I. Verify each identity by graphing the corresponding functions on the graphing calculator.

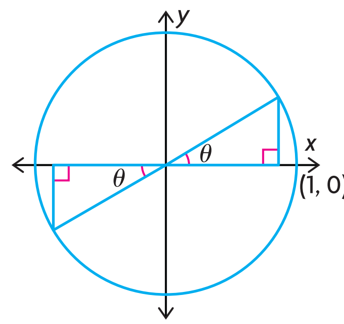
Creating equivalent expressions using the principal and related angles

L. Explain how you can tell, from this diagram of a unit circle, that
  i) $\sin(\pi - \theta) = \sin\theta$
  ii) $\cos(\pi - \theta) = -\cos\theta$
  iii) $\tan(\pi - \theta) = -\tan\theta$

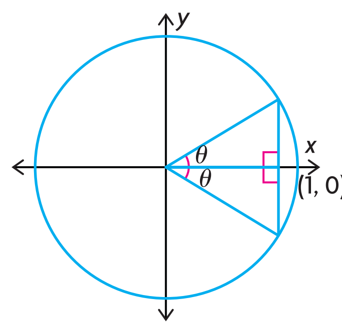


M. Write similar statements for the following diagrams.

i)



ii)



N. Summarize the strategies you used to identify and verify equivalent trigonometric expressions. Make a list of all the equivalent expressions you found.

Reflecting

O. How does a graphing calculator help you investigate the possible equivalence of two trigonometric expressions?

P. How can transformations be used to identify and confirm equivalent trigonometric expressions?

Q. How can the relationship between the acute angles in a right triangle be used to identify and confirm equivalent trigonometric expressions?

R. How can the relationship between a principal angle in standard position and the related acute angle be used to identify and confirm equivalent trigonometric expressions?

In Summary

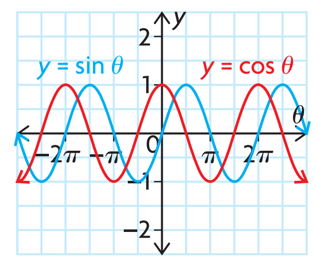
Key Ideas

- Because of their periodic nature, there are many equivalent trigonometric expressions.
- Two expressions may be equivalent if the graphs created by a graphing calculator of their corresponding functions coincide, producing only one visible graph over the entire domain of both functions. To demonstrate equivalency requires additional reasoning about the properties of both graphs.

Need to Know

- Horizontal translations that involve multiples of the period of a trigonometric function can be used to obtain two equivalent functions with the same graph. For example, the sine function has a period of 2π, so the graphs of $f(\theta) = \sin \theta$ and $f(\theta) = \sin (\theta + 2\pi)$ are the same. Therefore, $\sin \theta = \sin (\theta + 2\pi)$.
- Horizontal translations of $\frac{\pi}{2}$ that involve both a sine function and a cosine function can be used to obtain two equivalent functions with the same graph. Translating the cosine function $\frac{\pi}{2}$ to the right $\left(f(\theta) = \cos\left(\theta - \frac{\pi}{2}\right)\right)$ results in the graph of the sine function, $f(\theta) = \sin \theta$.

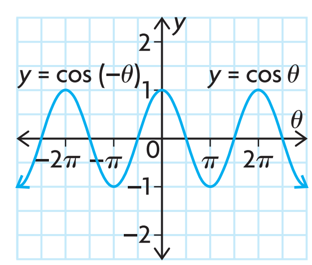
  Similarly, translating the sine function $\frac{\pi}{2}$ to the left $\left(f(\theta) = \sin\left(\theta + \frac{\pi}{2}\right)\right)$ results in the graph of the cosine function, $f(\theta) = \cos \theta$.



$\sin \theta = \cos\left(\theta - \frac{\pi}{2}\right)$

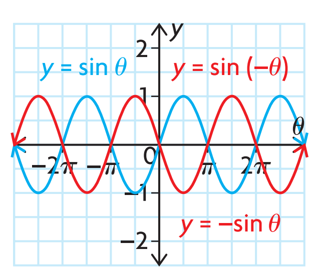
$\sin\left(\theta + \frac{\pi}{2}\right) = \cos \theta$

- Since $f(\theta) = \cos \theta$ is an even function, reflecting its graph across the y-axis results in two equivalent functions with the same graph.

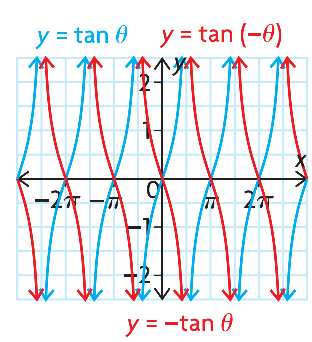


$\cos \theta = \cos(-\theta)$

- $f(\theta) = \sin \theta$ and $f(\theta) = \tan \theta$ are odd and have the property of rotational symmetry about the origin. Reflecting these functions across both the x-axis and the y-axis produces the same effect as rotating the function through 180° about the origin. Thus, the same graph is produced.



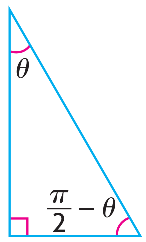
$\sin(-\theta) = -\sin \theta$



$\tan(-\theta) = -\tan \theta$

(continued)

- The cofunction identities describe trigonometric relationships between the complementary angles θ and $\left(\frac{\pi}{2} - \theta\right)$ in a right triangle.



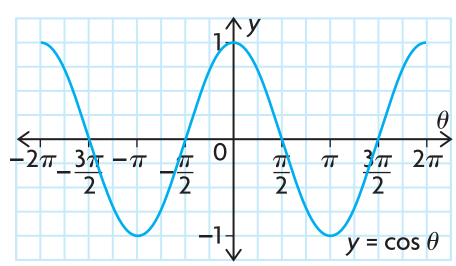
$$\sin \theta = \cos\left(\frac{\pi}{2} - \theta\right)$$
$$\cos \theta = \sin\left(\frac{\pi}{2} - \theta\right)$$
$$\tan \theta = \cot\left(\frac{\pi}{2} - \theta\right)$$

- You can identify equivalent trigonometric expressions by comparing principal angles drawn in standard position in quadrants II, III, and IV with their related acute angle, θ, in quadrant I.

Principal Angle in Quadrant II	Principal Angle in Quadrant III	Principal Angle in Quadrant IV
$\sin(\pi - \theta) = \sin\theta$	$\sin(\pi + \theta) = -\sin\theta$	$\sin(2\pi - \theta) = -\sin\theta$
$\cos(\pi - \theta) = -\cos\theta$	$\cos(\pi + \theta) = -\cos\theta$	$\cos(2\pi - \theta) = \cos\theta$
$\tan(\pi - \theta) = -\tan\theta$	$\tan(\pi + \theta) = \tan\theta$	$\tan(2\pi - \theta) = -\tan\theta$

FURTHER *Your Understanding*

1. **a)** Use transformations and the cosine function to write three equivalent expressions for the following graph.



b) Use transformations and a different trigonometric function to write three equivalent expressions for the graph.

2. **a)** Classify the reciprocal trigonometric functions as odd or even, and then write the corresponding equation.

b) Use transformations to explain why each equation is true.

3. Use the cofunction identities to write an expression that is equivalent to each of the following expressions.

a) $\sin\frac{\pi}{6}$ **c)** $\tan\frac{3\pi}{8}$ **e)** $\sin\frac{\pi}{8}$

b) $\cos\frac{5\pi}{12}$ **d)** $\cos\frac{5\pi}{16}$ **f)** $\tan\frac{\pi}{6}$

4. a) Write the cofunction identities for the reciprocal trigonometric functions.
b) Use transformations to explain why each identity is true.

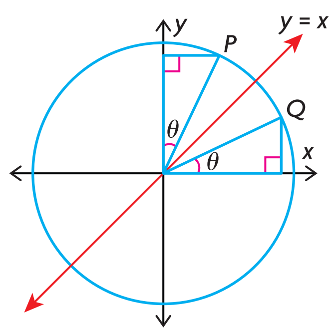
5. Write an expression that is equivalent to each of the following expressions, using the related acute angle.

a) $\sin \frac{7\pi}{8}$ **c)** $\tan \frac{5\pi}{4}$ **e)** $\sin \frac{13\pi}{8}$

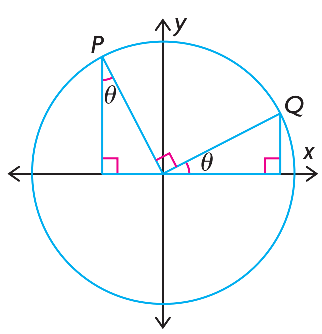
b) $\cos \frac{13\pi}{12}$ **d)** $\cos \frac{11\pi}{6}$ **f)** $\tan \frac{5\pi}{3}$

6. Show that each equation is true, using the given diagram.

a) $\cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta$



b) $\cos\left(\frac{\pi}{2} + \theta\right) = -\sin \theta$



7. State whether each of the following are true or false. For those that are false, justify your decision.

a) $\cos(\theta + 2\pi) = \cos \theta$ **d)** $\tan(\pi - \theta) = \tan \theta$

b) $\sin(\pi - \theta) = -\sin \theta$ **e)** $\cot\left(\frac{\pi}{2} + \theta\right) = \tan \theta$

c) $\cos \theta = -\cos(\theta + 4\pi)$ **f)** $\sin(\theta + 2\pi) = \sin(-\theta)$