7.2 Compound Angle Formulas

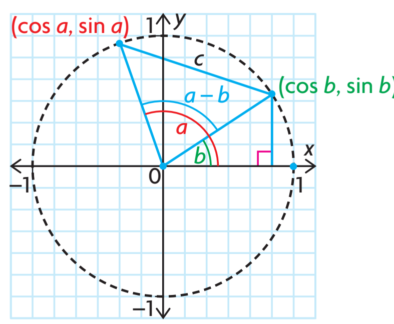
GOAL

Verify and use compound angle formulas.

INVESTIGATE the Math

compound angle
an angle that is created by adding or subtracting two or more angles

The cosine of the compound angle $(a - b)$ can be expressed in terms of the sines and cosines of a and b. Consider the following unit circle diagram:



By the cosine law, $c^2 = 1^2 + 1^2 - 2(1)(1)\cos(a - b)$

① $c^2 = 2 - 2\cos(a - b)$

However, c has endpoints of $(\cos a, \sin a)$ and $(\cos b, \sin b)$.

By the distance formula, $c = \sqrt{(\sin a - \sin b)^2 + (\cos a - \cos b)^2}$

Squaring both sides,

$c^2 = (\sin a - \sin b)^2 + (\cos a - \cos b)^2$

$c^2 = \sin^2 a - 2\sin a \sin b + \sin^2 b + \cos^2 a - 2\cos a \cos b + \cos^2 b$

$c^2 = \sin^2 a + \cos^2 a - 2\sin a \sin b - 2\cos a \cos b + \sin^2 b + \cos^2 b$

$c^2 = 1 - 2\sin a \sin b - 2\cos a \cos b + 1$

② $c^2 = 2 - 2\sin a \sin b - 2\cos a \cos b$

Equating ① and ②,

$2 - 2\cos(a - b) = 2 - 2\sin a \sin b - 2\cos a \cos b$

Solving for $\cos(a - b)$,

$\cos(a - b) = \sin a \sin b + \cos a \cos b$

? How can other formulas be developed to relate the primary trigonometric ratios of a compound angle to the trigonometric ratios of each angle in the compound angle?

A. Use a calculator and the special triangles to verify that the subtraction formula for cosine works if $a = 45°$ and $b = 30°$. Repeat for $a = \frac{\pi}{3}$ and $b = \frac{\pi}{6}$.

B. Use the subtraction formula for cosine to obtain an addition formula for cosine, $\cos(a + b)$, as follows:
- **i)** Rewrite the compound angle equation for $\cos(a - b)$.
- **ii)** Replace b with $(-b)$, and derive an equation for $\cos(a + b)$.
- **iii)** Simplify this equation, using your knowledge of even and odd functions, to write $\sin(-b)$ in terms of $\sin b$, and $\cos(-b)$ in terms of $\cos b$.

C. Use a calculator and the special triangles to verify your addition formula for cosine if $a = \frac{\pi}{3}$ and $b = \frac{\pi}{4}$.

D. To find an addition formula for sine, $\sin(a + b)$, use the cofunction identity $\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$.
- **i)** Write $\sin(a + b) = \cos\left(\frac{\pi}{2} - (a + b)\right) = \cos\left(\left(\frac{\pi}{2} - a\right) - b\right)$.
- **ii)** Use the subtraction formula for cosine to expand and simplify this formula.

E. Use a calculator and the special triangles to verify your addition formula for sine by substituting $a = \frac{\pi}{3}$ and $b = \frac{\pi}{4}$.

F. Determine and verify a subtraction formula for sine, $\sin(a - b)$, using the addition formula you found in part D and the strategy you used in part B.

G. Recall that $\tan\theta = \frac{\sin\theta}{\cos\theta}$. Use this identity to determine addition and subtraction formulas for $\tan(a + b)$ and $\tan(a - b)$. Use a calculator and the special triangles to verify your formulas if $a = \frac{\pi}{6}$ and $b = \frac{\pi}{4}$.

H. Make a list of all the compound angle formulas that you determined.

Reflecting

I. How did you use equivalent trigonometric expressions to simplify formulas in parts B, D, F, and G?

J. How did you use the special triangles to verify the addition and subtraction formulas you determined?

APPLY the Math

EXAMPLE 1 Selecting a strategy to determine the exact value of a trigonometric ratio

Determine the exact value of

a) $\cos(15°)$ **b)** $\tan\left(-\frac{5\pi}{12}\right)$

Solution

a) $\cos(15°)$

$= (\cos 45° - 30°)$

15° = 45° − 30°, so 15° can be expressed as the compound angle (45° − 30°).

$\cos(a - b)$

$= (\cos a)(\cos b) + (\sin a)(\sin b)$

$= (\cos 45°)(\cos 30°) + (\sin 45°)(\sin 30°)$

$= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right)$

$= \frac{\sqrt{3} + 1}{2\sqrt{2}}$

Use the subtraction formula for cosine to expand this expression where $a = 45°$ and $b = 30°$. Then use the special triangles to evaluate it.

b) $\tan\left(-\frac{5\pi}{12}\right)$

$= \tan\left(-\frac{\pi}{4} - \frac{\pi}{6}\right)$

$-\frac{5\pi}{12} = \frac{-5(180°)}{12} = -75°$

$-75° = -45° - 30°$

So $-\frac{5\pi}{12}$ can be expressed as the compound angle $\left(-\frac{\pi}{4} - \frac{\pi}{6}\right)$.

$\tan(a - b)$

$= \frac{\tan a - \tan b}{1 + \tan a \tan b}$

$= \frac{\tan\left(-\frac{\pi}{4}\right) - \tan\left(\frac{\pi}{6}\right)}{1 + \tan\left(-\frac{\pi}{4}\right)\tan\left(\frac{\pi}{6}\right)}$

Use the subtraction formula for tangent to expand this expression where $a = -\frac{\pi}{4}$ and $b = \frac{\pi}{6}$. Then use the special triangles to evaluate it.

$= \frac{-1 - \frac{1}{\sqrt{3}}}{1 + (-1)\left(\frac{1}{\sqrt{3}}\right)}$

Simplify.

$= \frac{\frac{-\sqrt{3} - 1}{\sqrt{3}}}{\frac{\sqrt{3} - 1}{\sqrt{3}}}$

Divide by multiplying by the reciprocal.

$= \frac{-\sqrt{3} - 1}{\sqrt{3} - 1}$

Compound angle formulas can be used, both forward and backward, to evaluate and simplify trigonometric expressions.

EXAMPLE 2 Using compound angle formulas to simplify trigonometric expressions

Simplify each expression.

a) $\cos\frac{7\pi}{12}\cos\frac{5\pi}{12} + \sin\frac{7\pi}{12}\sin\frac{5\pi}{12}$

b) $\sin 2x \cos x - \cos 2x \sin x$

Solution

a) $\cos(a - b)$

$= (\cos a)(\cos b) + (\sin a)(\sin b)$ ← The expression given is the right side of the subtraction formula for cosine, where $a = \frac{7\pi}{12}$ and $b = \frac{5\pi}{12}$.

$\cos\frac{7\pi}{12}\cos\frac{5\pi}{12} + \sin\frac{7\pi}{12}\sin\frac{5\pi}{12}$

$= \cos\left(\frac{7\pi}{12} - \frac{5\pi}{12}\right)$

$= \cos\frac{\pi}{6}$ ← $\frac{7\pi}{12} - \frac{5\pi}{12} = \frac{2\pi}{12} = \frac{\pi}{6}$. Use a special triangle to evaluate $\cos\frac{\pi}{6}$.

$= \frac{\sqrt{3}}{2}$

b) $\sin(a - b)$

$= (\sin a)(\cos b) - (\cos a)(\sin b)$ ← The expression given is the right side of the subtraction formula for sine, where $a = 2x$ and $b = x$.

$\sin 2x \cos x - \cos 2x \sin x$

$= \sin(2x - x)$

$= \sin x$

By expressing an angle as a sum or difference of angles in the special triangles, exact values of other angles can be determined.

EXAMPLE 3 Calculating trigonometric ratios of compound angles

Evaluate $\sin(a + b)$, where a and b are obtuse angles; $\sin a = \frac{3}{5}$ and $\sin b = \frac{5}{13}$.

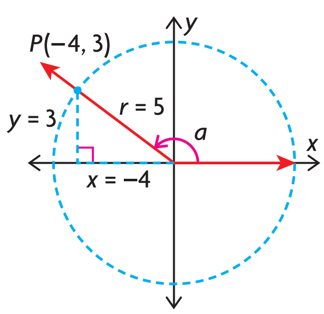
Solution

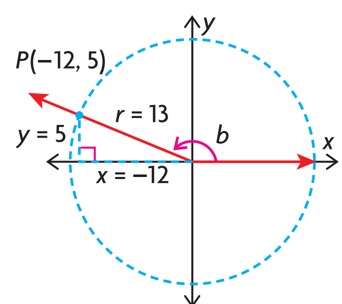
$$\sin a = \frac{3}{5} = \frac{y}{r} \qquad \text{and} \qquad \sin b = \frac{5}{13} = \frac{y}{r}$$

$$\begin{aligned} x^2 + y^2 &= r^2 \\ x^2 + 3^2 &= 5^2 \\ x^2 &= 25 - 9 \\ x &= \pm\sqrt{16} \\ x &= -4 \end{aligned} \qquad \begin{aligned} x^2 + y^2 &= r^2 \\ x^2 + 5^2 &= 13^2 \\ x^2 &= 169 - 25 \\ x &= \pm\sqrt{144} \\ x &= -12 \end{aligned}$$

Use the Pythagorean theorem to determine the x-coordinate of each point on the terminal arm. Since a and b are obtuse angles, their terminal arms lie in the second quadrant, where $\frac{\pi}{2} < a < \pi$ and $\frac{\pi}{2} < b < \pi$. In the second quadrant, x must be negative.

Sketch each angle in standard position.





$$\cos a = \frac{x}{r} = -\frac{4}{5} \qquad \cos b = \frac{x}{r} = -\frac{12}{13}$$

To determine $\sin(a + b)$, the sine and cosine of both a and b are required. Determine the cosine of a and b.

$$\begin{aligned} \sin(a + b) &= (\sin a)(\cos b) + (\cos a)(\sin b) \\ &= \left(\frac{3}{5}\right)\left(-\frac{12}{13}\right) + \left(-\frac{4}{5}\right)\left(\frac{5}{13}\right) \\ &= -\frac{36}{65} - \frac{20}{65} \\ &= -\frac{56}{65} \end{aligned}$$

Substitute the required trigonometric ratios into the compound angle formula for $\sin(a + b)$, and then evaluate.

Compound angle formulas can also be used to prove the equivalence of trigonometric expressions.

EXAMPLE 4 Identifying equivalent trigonometric expressions using compound angle formulas

Use compound angle formulas to show that $\sin(x - \pi)$, $\sin(x + \pi)$, and $\cos\left(x + \frac{\pi}{2}\right)$ are equivalent trigonometric expressions.

Solution

$$\begin{aligned}\sin(x - \pi) &= (\sin x)(\cos \pi) - (\cos x)(\sin \pi)\\ &= (\sin x)(-1) - (\cos x)(0)\\ &= -\sin x\end{aligned}$$

Use the subtraction formula for sine.

$$\begin{aligned}\sin(x + \pi) &= (\sin x)(\cos \pi) + (\cos x)(\sin \pi)\\ &= (\sin x)(-1) + (\cos x)(0)\\ &= -\sin x\end{aligned}$$

Use the addition formula for sine.

$$\begin{aligned}\cos\left(x + \frac{\pi}{2}\right) &= (\cos x)\left(\cos\frac{\pi}{2}\right) - (\sin x)\left(\sin\frac{\pi}{2}\right)\\ &= (\cos x)(0) - (\sin x)(1)\\ &= -\sin x\end{aligned}$$

Use the addition formula for cosine.

$$\sin(x - \pi) = \sin(x + \pi) = \cos\left(x + \frac{\pi}{2}\right)$$

They are all equivalent to the same expression, $-\sin x$.

In Summary

Key Idea

- The trigonometric ratios of compound angles are related to the trigonometric ratios of their component angles by the following compound angle formulas.

Addition Formulas	Subtraction Formulas
$\sin(a + b) = \sin a \cos b + \cos a \sin b$	$\sin(a - b) = \sin a \cos b - \cos a \sin b$
$\cos(a + b) = \cos a \cos b - \sin a \sin b$	$\cos(a - b) = \cos a \cos b + \sin a \sin b$
$\tan(a + b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$	$\tan(a - b) = \frac{\tan a - \tan b}{1 + \tan a \tan b}$

Need to Know

- You can use compound angle formulas to obtain exact values for trigonometric ratios.
- You can use compound angle formulas to show that some trigonometric expressions are equivalent.

CHECK Your Understanding

1. Rewrite each expression as a single trigonometric ratio.

a) $\sin a \cos 2a + \cos a \sin 2a$

b) $\cos 4x \cos 3x - \sin 4x \sin 3x$

2. Rewrite each expression as a single trigonometric ratio, and then evaluate the ratio.

a) $\dfrac{\tan 170° - \tan 110°}{1 + \tan 170° \tan 110°}$

b) $\cos \dfrac{5\pi}{12} \cos \dfrac{\pi}{12} + \sin \dfrac{5\pi}{12} \sin \dfrac{\pi}{12}$

3. Express each angle as a compound angle, using a pair of angles from the special triangles.

a) $75°$ **c)** $-\dfrac{\pi}{6}$ **e)** $105°$

b) $-15°$ **d)** $\dfrac{\pi}{12}$ **f)** $\dfrac{5\pi}{6}$

4. Determine the exact value of each trigonometric ratio.

a) $\sin 75°$ **c)** $\tan \dfrac{5\pi}{12}$ **e)** $\cos 105°$

b) $\cos 15°$ **d)** $\sin\left(-\dfrac{\pi}{12}\right)$ **f)** $\tan \dfrac{23\pi}{12}$

PRACTISING

5. K Use the appropriate compound angle formula to determine the exact value of each expression.

a) $\sin\left(\pi + \dfrac{\pi}{6}\right)$ **c)** $\tan\left(\dfrac{\pi}{4} + \pi\right)$ **e)** $\tan\left(\dfrac{\pi}{3} - \dfrac{\pi}{6}\right)$

b) $\cos\left(\pi - \dfrac{\pi}{4}\right)$ **d)** $\sin\left(-\dfrac{\pi}{2} + \dfrac{\pi}{3}\right)$ **f)** $\cos\left(\dfrac{\pi}{2} + \dfrac{\pi}{3}\right)$

6. Use the appropriate compound angle formula to create an equivalent expression.

a) $\sin(\pi + x)$ **c)** $\cos\left(x + \dfrac{\pi}{2}\right)$ **e)** $\sin(x - \pi)$

b) $\cos\left(x + \dfrac{3\pi}{2}\right)$ **d)** $\tan(x + \pi)$ **f)** $\tan(2\pi - x)$

7. Use transformations to explain why each expression you created in question 6 is equivalent to the given expression.

8. Determine the exact value of each trigonometric ratio.

a) $\cos 75°$
b) $\tan(-15°)$
c) $\cos \frac{11\pi}{12}$
d) $\sin \frac{13\pi}{12}$
e) $\tan \frac{7\pi}{12}$
f) $\tan \frac{-5\pi}{12}$

9. If $\sin x = \frac{4}{5}$ and $\sin y = -\frac{12}{13}$, $0 < x < \frac{\pi}{2}$, $\frac{3\pi}{2} < y < 2\pi$, evaluate

a) $\cos(x + y)$
b) $\sin(x + y)$
c) $\cos(x - y)$
d) $\sin(x - y)$
e) $\tan(x + y)$
f) $\tan(x - y)$

10. A α and β are acute angles in quadrant I, with $\sin \alpha = \frac{7}{25}$ and $\cos \beta = \frac{5}{13}$. Without using a calculator, determine the values of $\sin(\alpha + \beta)$ and $\tan(\alpha + \beta)$.

11. Use compound angle formulas to verify each of the following cofunction identities.

a) $\sin x = \cos\left(\frac{\pi}{2} - x\right)$
b) $\cos x = \sin\left(\frac{\pi}{2} - x\right)$

12. Simplify each expression.

a) $\sin(\pi + x) + \sin(\pi - x)$
b) $\cos\left(x + \frac{\pi}{3}\right) - \sin\left(x + \frac{\pi}{6}\right)$

13. T Simplify $\frac{\sin(f + g) + \sin(f - g)}{\cos(f + g) + \cos(f - g)}$.

14. C Create a flow chart to show how you would evaluate $\cos(a + b)$, given the values of $\sin a$ and $\sin b$, if both a and $b \in \left[0, \frac{\pi}{2}\right]$.

15. List the compound angle formulas you used in this lesson, and look for similarities and differences. Explain how you can use these similarities and differences to help you remember the formulas.

Extending

16. Prove $\sin C + \sin D = 2 \sin\left(\frac{C + D}{2}\right) \cos\left(\frac{C - D}{2}\right)$.

17. Determine $\cot(x + y)$ in terms of $\cot x$ and $\cot y$.

18. Prove $\cos C + \cos D = 2 \cos\left(\frac{C + D}{2}\right) \cos\left(\frac{C - D}{2}\right)$.

19. Prove $\cos C - \cos D = -2 \sin\left(\frac{C + D}{2}\right) \sin\left(\frac{C - D}{2}\right)$.