7.3 Double Angle Formulas

YOU WILL NEED

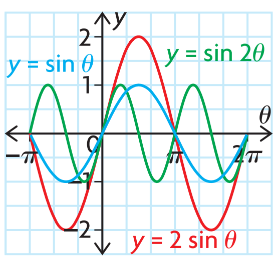
- graphing calculator

GOAL

Develop and use double angle formulas.

INVESTIGATE the Math

From your work with graphs of trigonometric functions, you already know that $f(\theta) = \sin 2\theta$ is not the same as $f(\theta) = 2 \sin \theta$.



$f(\theta) = \sin 2\theta$ is the graph of $y = \sin \theta$ compressed horizontally by a factor of $\frac{1}{2}$.

$f(\theta) = 2 \sin \theta$ is the graph of $y = \sin \theta$ stretched vertically by a factor of 2.

? How are the trigonometric ratios of an angle that has been doubled to 2θ related to the trigonometric ratios of the original angle θ?

A. Given $\sin 2\theta = \sin (\theta + \theta)$, use the appropriate compound angle formula to expand $\sin (\theta + \theta)$. Simplify both sides to develop a formula for $\sin 2\theta$.

B. Verify your double angle formula for sine by graphing each side as a function on a graphing calculator and examining the tables of values.

C. Verify that your double angle formula for sine works by evaluating both sides of the formula for $\theta = 45°$. Repeat for $\theta = \frac{\pi}{6}$.

D. Repeat parts A to C to develop a double angle formula for $\cos 2\theta$.

E. Use the identity $\sin^2 \theta + \cos^2 \theta = 1$ to eliminate $\sin \theta$ from the right side of your formula in part D. Verify that your new formula is correct by graphing and by substitution, as before.

F. Repeat part E, but this time eliminate $\cos\theta$ on the right side to develop an equivalent expression in terms of $\sin\theta$.

G. Repeat parts A to C to develop a double angle formula for $\tan 2\theta$.

H. Make a list of all the double angle formulas you developed.

Reflecting

I. How did you use compound angle formulas to develop double angle formulas?

J. Why were you able to develop three different formulas for $\cos 2\theta$?

K. How might you develop formulas for $\sin\frac{\theta}{2}$ and $\cos\frac{\theta}{2}$?

APPLY the Math

EXAMPLE 1 Using double angle formulas to simplify and evaluate expressions

Simplify each of the following expressions and then evaluate.

a) $2\sin\frac{\pi}{8}\cos\frac{\pi}{8}$

b) $\dfrac{2\tan\frac{\pi}{6}}{1-\tan^2\frac{\pi}{6}}$

Solution

a)

$$2\sin x\cos x = \sin 2x$$

$$2\sin\frac{\pi}{8}\cos\frac{\pi}{8} = \sin 2\left(\frac{\pi}{8}\right)$$

$$= \sin\frac{\pi}{4}$$

$$= \frac{1}{\sqrt{2}}$$

This expression is the right side of the double angle formula for sine.

In this expression, $x = \frac{\pi}{8}$.

Use the special triangles to evaluate.

b)

$$\frac{2\tan x}{1-\tan^2 x} = \tan 2x, \text{ where } \tan x \neq \pm 1$$

$$\frac{2\tan\frac{\pi}{6}}{1-\tan^2\frac{\pi}{6}} = \tan 2\left(\frac{\pi}{6}\right)$$

This expression is similar to the right side of the double angle formula for tangent. In this expression, $x = \frac{\pi}{6}$.

$$= \tan\frac{\pi}{3}$$

$$= \sqrt{3}$$

Use the special triangles to evaluate $\tan\frac{\pi}{3}$.

If you know one of the primary trigonometric ratios for any angle, then you can determine the other two. You can then determine the primary trigonometric ratios for this angle doubled.

EXAMPLE 2 Selecting a strategy to determine the value of trigonometric ratios for a double angle

If $\cos \theta = -\frac{2}{3}$ and $0 \leq \theta \leq 2\pi$, determine the value of $\cos 2\theta$ and $\sin 2\theta$.

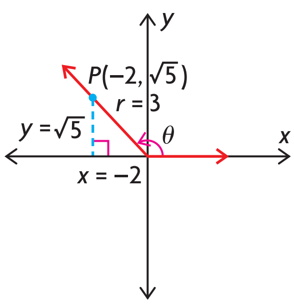
Solution

$\cos 2\theta = 2\cos^2 \theta - 1$ ← There are three double angle formulas for cosine. Since the value of $\cos \theta$ is given, choose the formula that is strictly in terms of cosine.

$= 2\left(-\frac{2}{3}\right)^2 - 1$ ← Let $\cos \theta = -\frac{2}{3}$ and evaluate.

$= 2\left(\frac{4}{9}\right) - 1$

$= -\frac{1}{9}$



If θ is in quadrant II and $\cos \theta = \frac{x}{r} = -\frac{2}{3}$, then

$x^2 + y^2 = r^2$ ← Since cosine is negative, the terminal arm of θ can lie in quadrant II or quadrant III. Since $r > 0$, x must be negative. Use the Pythagorean theorem to calculate y.

$(-2)^2 + y^2 = 3^2$

$4 + y^2 = 9$

$y^2 = \pm 5$ ← In quadrant II, y is positive.

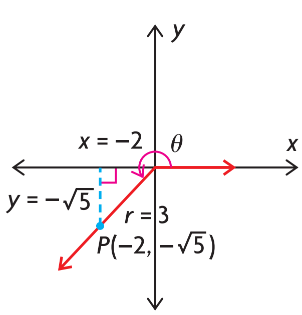
$y = \sqrt{5}$

So, $\sin \theta = \frac{y}{r} = \frac{\sqrt{5}}{3}$

$\sin 2\theta = 2 \sin \theta \cos \theta$

$= 2\left(\frac{\sqrt{5}}{3}\right)\left(-\frac{2}{3}\right)$ ← Use the double angle formula for sine, and replace $\sin \theta$ and $\cos \theta$ with the values calculated.

$= -\frac{4\sqrt{5}}{9}$



If θ is in quadrant III, $y = -\sqrt{5}$. ← Using the value of y that was calculated above, y is negative in quadrant III.

So $\sin \theta = \frac{y}{r} = \frac{-\sqrt{5}}{3}$

$\sin 2\theta = 2 \sin \theta \cos \theta$ ← Use the double angle formula for sine, and replace $\sin \theta$ and $\cos \theta$ with the values calculated.

$= 2\left(\frac{-\sqrt{5}}{3}\right)\left(-\frac{2}{3}\right)$

$= \frac{4\sqrt{5}}{9}$

EXAMPLE 3 Selecting a strategy to determine the primary trigonometric ratios for a double angle

Given $\tan\theta = -\frac{3}{4}$, where $\frac{3\pi}{2} \le \theta \le 2\pi$, calculate the value of $\cos 2\theta$.

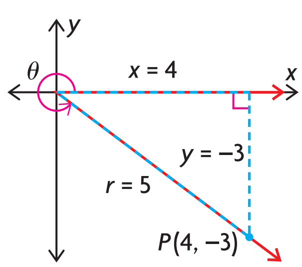
Solution

$$\begin{aligned} \tan\theta &= \frac{y}{x} = \frac{-3}{4} \\ x^2 + y^2 &= r^2 \\ 4^2 + (-3)^2 &= r^2 \\ 16 + 9 &= r^2 \\ \pm\sqrt{25} &= r \\ 5 &= r \end{aligned}$$

Since $\frac{3\pi}{2} \le \theta \le 2\pi$, the terminal arm of the angle lies in quadrant IV. Therefore, x is positive and y is negative. Use the Pythagorean theorem to determine r.

Since r is always positive, $r > 0$.

Draw θ in standard position.



$\sin\theta = \frac{y}{r} = \frac{-3}{5}$ and $\cos\theta = \frac{x}{r} = \frac{4}{5}$

Since $\cos 2\theta = \cos^2\theta - \sin^2\theta$, determine the values of $\sin\theta$ and $\cos\theta$.

$$\begin{aligned} \cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &= \left(\frac{4}{5}\right)^2 - \left(\frac{-3}{5}\right)^2 \\ &= \frac{16}{25} - \frac{9}{25} \\ &= \frac{7}{25} \end{aligned}$$

Use one of the double angle formulas for $\cos 2\theta$, and substitute the values of $\sin\theta$ and $\cos\theta$.

The double angle formulas can be used to create other equivalent trigonometric relationships.

EXAMPLE 4 Using reasoning to derive other formulas from the double angle formulas

Develop a formula for $\sin \frac{x}{2}$.

Solution

$$\cos 2x = 1 - 2\sin^2 x$$
$$\cos 2\left(\frac{x}{2}\right) = 1 - 2\sin^2\left(\frac{x}{2}\right)$$

Since $\cos x = \cos 2\left(\frac{x}{2}\right)$, replace x with $\frac{x}{2}$ in the cosine double angle formula that only involves sine.

$$\cos x = 1 - 2\sin^2\left(\frac{x}{2}\right)$$
$$2\sin^2\left(\frac{x}{2}\right) = 1 - \cos x$$
$$\sin^2\left(\frac{x}{2}\right) = \frac{1 - \cos x}{2}$$
$$\sin\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1 - \cos x}{2}}$$

Solve for $\sin\left(\frac{x}{2}\right)$ as follows:

- Add $2\sin^2\left(\frac{x}{2}\right)$ to both sides.
- Subtract $\cos x$ from both sides.
- Divide both sides by 2.
- Take the square root of both sides.

In Summary

Key Idea

- The double angle formulas show how the trigonometric ratios for a double angle, 2θ, are related to the trigonometric ratios for the original angle, θ.

Double Angle Formula for Sine

$\sin 2\theta = 2\sin\theta\cos\theta$

Double Angle Formulas for Cosine

$\cos 2\theta = \cos^2\theta - \sin^2\theta$
$\cos 2\theta = 2\cos^2\theta - 1$
$\cos 2\theta = 1 - 2\sin^2\theta$

Double Angle Formula for Tangent

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

Need to Know

- The double angle formulas can be derived from the appropriate compound angle formulas.
- You can use the double angle formulas to simplify expressions and to calculate exact values.
- The double angle formulas can be used to develop other equivalent formulas.

CHECK Your Understanding

1. Express each of the following as a single trigonometric ratio.

a) $2 \sin 5x \cos 5x$

b) $\cos^2 \theta - \sin^2 \theta$

c) $1 - 2 \sin^2 3x$

d) $\dfrac{2 \tan 4x}{1 - \tan^2 4x}$

e) $4 \sin \theta \cos \theta$

f) $2 \cos^2 \dfrac{\theta}{2} - 1$

2. Express each of the following as a single trigonometric ratio and then evaluate.

a) $2 \sin 45° \cos 45°$

b) $\cos^2 30° - \sin^2 30°$

c) $2 \sin \dfrac{\pi}{12} \cos \dfrac{\pi}{12}$

d) $\cos^2 \dfrac{\pi}{12} - \sin^2 \dfrac{\pi}{12}$

e) $1 - 2 \sin^2 \dfrac{3\pi}{8}$

f) $2 \tan 60° \cos^2 60°$

3. Use a double angle formula to rewrite each trigonometric ratio.

a) $\sin 4\theta$

b) $\cos 3x$

c) $\tan x$

d) $\cos 6\theta$

e) $\sin x$

f) $\tan 5\theta$

PRACTISING

4. **K** Determine the values of $\sin 2\theta$, $\cos 2\theta$, and $\tan 2\theta$, given $\cos \theta = \frac{3}{5}$ and $0 \leq \theta \leq \frac{\pi}{2}$.

5. Determine the values of $\sin 2\theta$, $\cos 2\theta$, and $\tan 2\theta$, given $\tan \theta = -\frac{7}{24}$ and $\frac{\pi}{2} \leq \theta \leq \pi$.

6. Determine the values of $\sin 2\theta$, $\cos 2\theta$, and $\tan 2\theta$, given $\sin \theta = -\frac{12}{13}$ and $\frac{3\pi}{2} \leq \theta \leq 2\pi$.

7. Determine the values of $\sin 2\theta$, $\cos 2\theta$, and $\tan 2\theta$, given $\cos \theta = -\frac{4}{5}$ and $\frac{\pi}{2} \leq \theta \leq \pi$.

8. **A** Determine the value of a in the following equation: $2 \tan x - \tan 2x + 2a = 1 - \tan 2x \tan^2 x$.

9. Jim needs to find the sine of $\frac{\pi}{8}$. If he knows that $\cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$, how can he use this fact to find the sine of $\frac{\pi}{8}$? What is his answer?

10. Marion needs to find the cosine of $\frac{\pi}{12}$. If she knows that $\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$, how can she use this fact to find the cosine of $\frac{\pi}{12}$? What is her answer?

11. T **a)** Use a double angle formula to develop a formula for $\sin 4x$ in terms of x.

b) Use the formula you developed in part a) to verify that $\sin \frac{2\pi}{3} = \sin \frac{8\pi}{3}$.

12. Use the appropriate compound angle formula and double angle formula to develop a formula for

a) $\sin 3\theta$ in terms of $\cos \theta$ and $\sin \theta$

b) $\cos 3\theta$ in terms of $\cos \theta$ and $\sin \theta$

c) $\tan 3\theta$ in terms of $\tan \theta$

13. The angle x lies in the interval $\frac{\pi}{2} \leq x \leq \pi$, and $\sin^2 x = \frac{8}{9}$. Without using a calculator, determine the value of

a) $\sin 2x$

b) $\cos 2x$

c) $\cos \frac{x}{2}$

d) $\sin 3x$

14. C Create a flow chart to show how you would evaluate $\sin 2a$, given the value of $\sin a$, if $a \in \left[\frac{\pi}{2}, \pi\right]$.

15. Describe how you could use your knowledge of double angle formulas to sketch the graph of each function. Include a sketch with your description.

a) $f(x) = \sin x \cos x$

b) $f(x) = 2\cos^2 x$

c) $f(x) = \dfrac{\tan x}{1 - \tan^2 x}$

Extending

16. Eliminate A from each pair of equations to find an equation that relates x to y.

a) $x = \tan 2A, y = \tan A$

b) $x = \cos 2A, y = \cos A$

c) $x = \cos 2A, y = \csc A$

d) $x = \sin 2A, y = \sec 4A$

17. Solve each equation for values of x in the interval $0 \leq x \leq 2\pi$.

a) $\cos 2x = \sin x$

b) $\sin 2x - 1 = \cos 2x$

18. Express each of the following in terms of $\tan \theta$.

a) $\sin 2\theta$

b) $\cos 2\theta$

c) $\dfrac{\sin 2\theta}{1 + \cos 2\theta}$

d) $\dfrac{1 - \cos 2\theta}{\sin 2\theta}$