7.4 Proving Trigonometric Identities

YOU WILL NEED

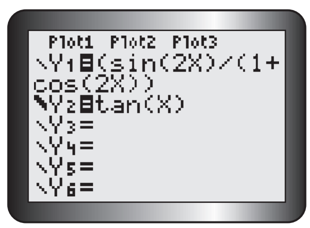
- graphing calculator

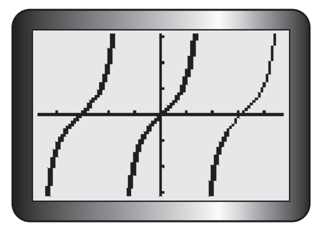
GOAL

Use equivalent trigonometric relationships to prove that an equation is an identity.

LEARN ABOUT the Math

When Alysia graphs the function $f(x) = \frac{\sin 2x}{1 + \cos 2x}$ using a graphing calculator, she sees that her graph looks the same as the graph for the tangent function $f(x) = \tan x$.





She makes a conjecture that $\frac{\sin 2x}{1 + \cos 2x} = \tan x$ is a trigonometric **identity**. In other words, she predicts that this equation is true for all values of x for which the expressions in the equation are defined.

? How can Alysia prove that her conjecture is true?

EXAMPLE 1 Using reasoning to prove an identity that involves double angles

Prove that $\frac{\sin 2x}{1 + \cos 2x} = \tan x$.

Solution

$$\begin{aligned} LS &= \frac{\sin 2x}{1 + \cos 2x} \\ &= \frac{2 \sin x \cos x}{1 + 2\cos^2 x - 1} \\ &= \frac{2 \sin x \cos x}{2 \cos^2 x} \\ &= \frac{\sin x}{\cos x} \\ &= \tan x = RS \end{aligned}$$

Begin with the left side (LS) because you can use double angle formulas to express the LS, using the same **argument** as the right side (RS).

After applying the double angle formulas, simplify the denominator. Then divide the numerator and the denominator by 2 cos x.

Since both sides are equal,

$$\frac{\sin 2x}{1 + \cos 2x} = \tan x$$

The expressions are equivalent for all real numbers, except where $\cos 2x = -1$ and $\cos x = 0$.

Reflecting

A. Why was the left side of the identity simplified at the beginning of the solution?

B. Which formula for $\cos 2x$ was used, and why? Could another formula have been used instead?

C. If you replaced x with $\frac{\pi}{4}$ in Alysia's conjecture and you showed that both sides result in the same value, could you conclude that the equation is an identity? Explain.

APPLY the Math

EXAMPLE 2 Proving that an equation is not an identity

Prove that $\sin x + \sin 2x = \sin 3x$ is not an identity.

Solution

Let $x = \frac{\pi}{2}$.

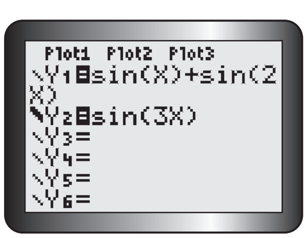
Choose any value for which both sides of the equation are defined, and evaluate both sides.

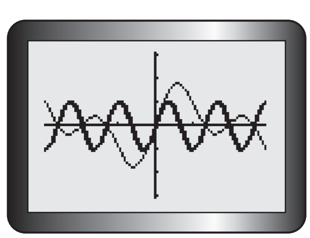
$$\begin{aligned} \text{LS} &= \sin\left(\frac{\pi}{2}\right) + \sin 2\left(\frac{\pi}{2}\right) \\ &= 1 + 0 \\ &= 1 \end{aligned}$$

$$\begin{aligned} \text{RS} &= \sin 3\left(\frac{\pi}{2}\right) \\ &= -1 \end{aligned}$$

Since there is a value for which the left side does not equal the right side, the equation is not an identity.

$x = \frac{\pi}{2}$ is a **counterexample**—it disproves the equivalence of both sides of the equation.





Graphing both sides of the equation results in very different graphs.

EXAMPLE 3 Using reasoning to prove a cofunction identity

Prove that $\cos\left(\frac{\pi}{2} + x\right) = -\sin x$.

Solution

$$\begin{aligned} \text{LS} &= \cos\left(\frac{\pi}{2} + x\right) \\ &= \cos\left(\frac{\pi}{2}\right)\cos x - \sin\left(\frac{\pi}{2}\right)\sin x \\ &= (0)\cos x - (1)\sin x \\ &= 0 - \sin x \\ &= -\sin x \\ &= \text{RS} \end{aligned}$$

Begin with the left side because a compound angle formula can be used to simplify the expression on the left side. Substitute the numerical values of $\cos\left(\frac{\pi}{2}\right)$ and $\sin\left(\frac{\pi}{2}\right)$.

Since both sides are equal,

$$\cos\left(\frac{\pi}{2} + x\right) = -\sin x$$

Because there is no denominator or square root on either side of the equation, the expressions are equivalent for all real numbers.

When you encounter a more complicated identity, you may be able to use several different strategies to prove the equivalence of the expressions.

EXAMPLE 4 Using reasoning to prove an identity that involves rational trigonometric expressions

Prove that $\frac{\cos(x - y)}{\cos(x + y)} = \frac{1 + \tan x \tan y}{1 - \tan x \tan y}$.

Solution

$$\begin{aligned} \text{RS} &= \frac{1 + \tan x \tan y}{1 - \tan x \tan y} \\ &= \frac{1 + \left(\frac{\sin x}{\cos x}\right)\left(\frac{\sin y}{\cos y}\right)}{1 - \left(\frac{\sin x}{\cos x}\right)\left(\frac{\sin y}{\cos y}\right)} \times \frac{(\cos x)(\cos y)}{(\cos x)(\cos y)} \\ &= \frac{(\cos x)(\cos y) + (\sin x)(\sin y)}{(\cos x)(\cos y) - (\sin x)(\sin y)} \\ &= \frac{\cos(x - y)}{\cos(x + y)} \\ &= \text{LS} \end{aligned}$$

Start with the right side. Replace $\tan x$ with $\frac{\sin x}{\cos x}$, and replace $\tan y$ with $\frac{\sin y}{\cos y}$. Then multiply the expression by $\frac{(\cos x)(\cos y)}{(\cos x)(\cos y)}$ (because this equals 1) to get one numerator and one denominator.

Rewrite the expressions in the numerator and the denominator using compound angle formulas.

Since both sides are equal,

$$\frac{\cos(x - y)}{\cos(x + y)} = \frac{1 + \tan x \tan y}{1 - \tan x \tan y}$$

The expressions are equivalent for all real numbers, except where $\cos(x + y) = 0$ and $\tan x \tan y = 1$.

Sometimes, you may need to factor if you want to prove that a given equation is an identity.

EXAMPLE 5 Using a factoring strategy to prove an identity

Prove that $\tan 2x - 2 \tan 2x \sin^2 x = \sin 2x$.

Solution

$$\begin{aligned} \text{LS} &= \tan 2x - 2 \tan 2x \sin^2 x \\ &= \tan 2x(1 - 2 \sin^2 x) \\ &= \tan 2x \cos 2x \\ &= \frac{\sin 2x}{\cos 2x}(\cos 2x) \\ &= \sin 2x, \cos 2x \neq 0 \\ &= \text{RS} \end{aligned}$$

Begin with the more complicated side.

Factor $\tan 2x$ out of the two terms.

The expression inside the brackets can be simplified using a double angle formula.

Write $\tan 2x$ as $\frac{\sin 2x}{\cos 2x}$, and simplify the resulting expression.

Since both sides are equal,

$\tan 2x - 2 \tan 2x \sin^2 x = \sin 2x$, $\cos 2x \neq 0$.

The expressions are equivalent for all real numbers, except where $\cos 2x = 0$. The left side involves the tangent function, which was expressed as a quotient, so the denominator cannot be 0.

In Summary

Key Ideas

- A trigonometric identity states the equivalence of two trigonometric expressions. It is written as an equation that involves trigonometric ratios, and the solution set is all real numbers for which the expressions on both sides of the equation are defined. As a result, the equation has an infinite number of solutions.
- Some trigonometric identities are the result of a definition, while others are derived from relationships that exist among trigonometric ratios.

Need to Know

- The following trigonometric identities are important for you to remember:

Identities Based on Definitions

Reciprocal Identities

$$\csc x = \frac{1}{\sin x}$$

$$\sec x = \frac{1}{\cos x}$$

$$\cot x = \frac{1}{\tan x}$$

Identities Derived from Relationships

Quotient Identities

$$\tan x = \frac{\sin x}{\cos x}$$

$$\cot x = \frac{\cos x}{\sin x}$$

Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x$$

$$1 + \cot^2 x = \csc^2 x$$

Double Angle Formulas

$$\sin 2x = 2 \sin x \cos x$$

$$\begin{aligned} \cos 2x &= \cos^2 x - \sin^2 x \\ &= 2\cos^2 x - 1 \\ &= 1 - 2\sin^2 x \end{aligned}$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$$

Addition and Subtraction Formulas

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

$$\sin(x - y) = \sin x \cos y - \cos x \sin y$$

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

$$\cos(x - y) = \cos x \cos y + \sin x \sin y$$

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

$$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

- You can verify the truth of a given trigonometric identity by graphing each side separately and showing that the two graphs are the same.
- To prove that a given equation is an identity, the two sides of the equation must be shown to be equivalent. This can be accomplished using a variety of strategies, such as
  - simplifying the more complicated side until it is identical to the other side, ormanipulating both sides to get the same expression
  - rewriting expressions using any of the identities stated above
  - using a common denominator or factoring, where possible

CHECK Your Understanding

1. Jared claims that $\sin x = \cos x$ is an identity, since $\sin \frac{\pi}{4} = \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$. Use a counterexample to disprove his claim.

2. **a)** Use a graphing calculator to graph $f(x) = \sin x$ and $g(x) = \tan x \cos x$ for $-2\pi \le x \le 2\pi$.
b) Write a trigonometric identity based on your graphs.
c) Simplify one side of your identity to prove it is true.
d) This identity is true for all real numbers, except where $\cos x = 0$. Explain why.

3. Graph the appropriate functions to match each expression on the left with the equivalent expression on the right.

a)	$\sin x \cot x$	**A**	$\sin^2 x + \cos^2 x + \tan^2 x$
b)	$1 - 2\sin^2 x$	**B**	$1 + 2\sin x \cos x$
c)	$(\sin x + \cos x)^2$	**C**	$\cos x$
d)	$\sec^2 x$	**D**	$2\cos^2 x - 1$

4. Prove algebraically that the expressions you matched in question 3 are equivalent.

PRACTISING

5. **K** Give a counterexample to show that each equation is not an identity.

a) $\cos x = \frac{1}{\cos x}$

b) $1 - \tan^2 x = \sec^2 x$

c) $\sin(x + y) = \cos x \cos y + \sin x \sin y$

d) $\cos 2x = 1 + 2\sin^2 x$

6. **A** Graph the expression $\frac{1 - \tan^2 x}{1 + \tan^2 x}$, and make a conjecture about another expression that is equivalent to this expression.

7. Prove your conjecture in question 6.

8. Prove that $\frac{1 + \tan x}{1 + \cot x} = \frac{1 - \tan x}{\cot x - 1}$.

9. Prove each identity.

a) $\frac{\cos^2 \theta - \sin^2 \theta}{\cos^2 \theta + \sin \theta \cos \theta} = 1 - \tan \theta$

b) $\tan^2 x - \sin^2 x = \sin^2 x \tan^2 x$

c) $\tan^2 x - \cos^2 x = \frac{1}{\cos^2 x} - 1 - \cos^2 x$

d) $\frac{1}{1 + \cos \theta} + \frac{1}{1 - \cos \theta} = \frac{2}{\sin^2 \theta}$

10. Prove each identity.

a) $\cos x \tan^3 x = \sin x \tan^2 x$

b) $\sin^2 \theta + \cos^4 \theta = \cos^2 \theta + \sin^4 \theta$

c) $(\sin x + \cos x)\left(\dfrac{\tan^2 x + 1}{\tan x}\right) = \dfrac{1}{\cos x} + \dfrac{1}{\sin x}$

d) $\tan^2 \beta + \cos^2 \beta + \sin^2 \beta = \dfrac{1}{\cos^2 \beta}$

e) $\sin\left(\dfrac{\pi}{4} + x\right) + \sin\left(\dfrac{\pi}{4} - x\right) = \sqrt{2} \cos x$

f) $\sin\left(\dfrac{\pi}{2} - x\right) \cot\left(\dfrac{\pi}{2} + x\right) = -\sin x$

11. Prove each identity.

a) $\dfrac{\cos 2x + 1}{\sin 2x} = \cot x$

b) $\dfrac{\sin 2x}{1 - \cos 2x} = \cot x$

c) $(\sin x + \cos x)^2 = 1 + \sin 2x$

d) $\cos^4 \theta - \sin^4 \theta = \cos 2\theta$

e) $\cot \theta - \tan \theta = 2 \cot 2\theta$

f) $\cot \theta + \tan \theta = 2 \csc 2\theta$

g) $\dfrac{1 + \tan x}{1 - \tan x} = \tan\left(x + \dfrac{\pi}{4}\right)$

h) $\csc 2x + \cot 2x = \cot x$

i) $\dfrac{2 \tan x}{1 + \tan^2 x} = \sin 2x$

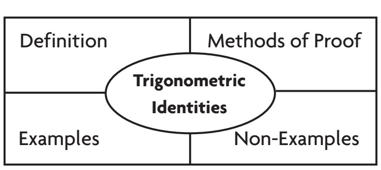
j) $\sec 2t = \dfrac{\csc t}{\csc t - 2 \sin t}$

k) $\csc 2\theta = \dfrac{1}{2}(\sec \theta)(\csc \theta)$

l) $\sec t = \dfrac{\sin 2t}{\sin t} - \dfrac{\cos 2t}{\cos t}$

12. Graph the expression $\dfrac{\sin x + \sin 2x}{1 + \cos x + \cos 2x}$, and make a conjecture about another expression that is equivalent to this expression.

13. Prove your conjecture in question 12.



14. Copy the chart shown, and complete it to summarize what you know about trigonometric identities.

15. Your friend wants to know whether the equation $2 \sin x \cos x = \cos 2x$ is an identity. Explain how she can determine whether it is an identity. If it is an identity, explain how she can prove this. If it is not an identity, explain how she can change one side of the equation to make it an identity.

Extending

16. Each of the following expressions can be written in the form $a \sin 2x + b \cos 2x + c$. Determine the values of a, b, and c.

a) $2 \cos^2 x + 4 \sin x \cos x$

b) $-2 \sin x \cos x - 4 \sin^2 x$

17. Express $8 \cos^4 x$ in the form $a \cos 4x + b \cos 2x + c$. State the values of the constants a, b, and c.