7.5 Solving Linear Trigonometric Equations

GOAL

Solve linear trigonometric equations algebraically and graphically.

YOU WILL NEED

- graphing calculator

LEARN ABOUT the Math

In Lesson 7.4, you learned how to prove that a given trigonometric equation is an identity. Not all trigonometric equations are identities, however. To see the difference between an equation that is an identity and an equation that is not, consider the following two equations on the domain $0 \le x \le 2\pi$: $\sin^2 x + \cos^2 x = 1$ and $2 \sin x - 1 = 0$.

The first equation is true for all values of x in the given domain, so it is an identity.

The second equation is true for only some values of x, so it is not an identity.

? How can you solve a trigonometric equation that is not an identity?

EXAMPLE 1 Selecting a strategy to determine the solutions for a linear trigonometric equation

You are given the equation $2 \sin x + 1 = 0$, $0 \le x \le 2\pi$.

a) Determine all the solutions in the specified interval.

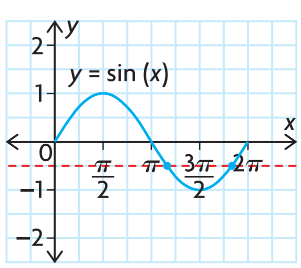
b) Verify the solutions using graphing technology.

Solution

a) $2 \sin x + 1 = 0$

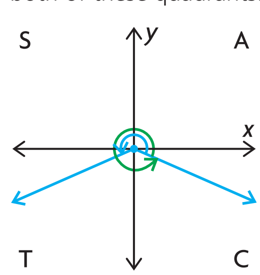
$2 \sin x = -1$

$\sin x = -\frac{1}{2}$



Two solutions are possible in the specified interval, $0 \le x \le 2\pi$, since the sine graph will complete one cycle in this interval.

Rearrange the equation to isolate $\sin x$.
Sketch a graph of the sine function to estimate where its value is $-\frac{1}{2}$.
From the graph, one solution is possible when $\pi \le x \le \frac{3\pi}{2}$ and another solution is possible when $\frac{3\pi}{2} \le x \le 2\pi$. Therefore, the terminal arms of the two angles lie in quadrants III and IV. This makes sense since r is positive and y is negative, so the sine ratio is negative for angles in both of these quadrants. This is confirmed by the CAST rule.



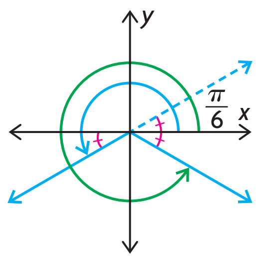
Determine the related acute angle.

$$\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

$\frac{\pi}{6}$ is a special angle.

Using the special triangle that contains $\frac{\pi}{6}$ and $\frac{\pi}{3}$, $\sin \frac{\pi}{6} = \frac{1}{2}$.

Use the related angle to determine the required solutions in the given interval.

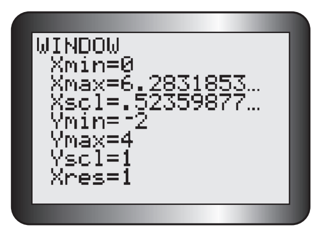


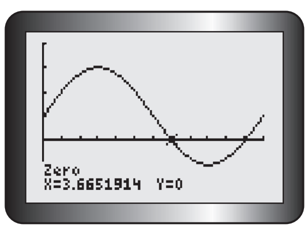
The solution in quadrant III is $\pi + \frac{\pi}{6} = \frac{7\pi}{6}$.

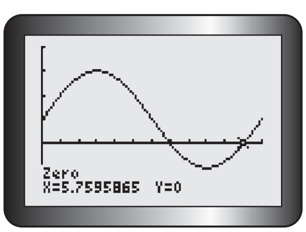
The solution in quadrant IV is $2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$.

b) Graph $f(x) = 2 \sin x + 1$ in radian mode, for $0 \le x \le 2\pi$, and determine the zeros.

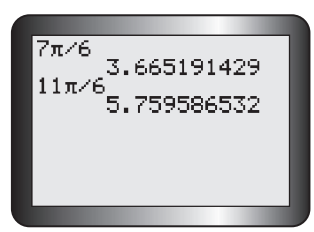
Use the window settings that match the domain for Xmin and Xmax. Use a scale of $\frac{\pi}{6}$.







To verify the solutions found in part a), express the solutions as decimals.



The zeros are located at approximately 3.665 191 4 and 5.759 586 5. These values are very close to $\frac{7\pi}{6}$ and $\frac{11\pi}{6}$.

Reflecting

A. How was solving the equation $2 \sin x + 1 = 0$ like solving the equation $2x + 1 = 0$? How was it different?

B. Once $\sin x$ was isolated in Example 1, how was the sign of the trigonometric ratio used to determine the quadrants in which the solutions were located?

C. The interval in Example 1 was $0 \le x \le 2\pi$. If the interval had been $x \in \mathbf{R}$, how many solutions would the equation have had? Explain.

APPLY the Math

EXAMPLE 2 Using an algebraic strategy to determine the approximate solutions for a linear trigonometric equation

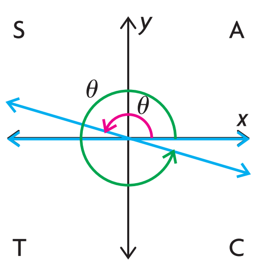
Solve $3(\tan\theta + 1) = 2$, where $0° \le \theta \le 360°$, correct to one decimal place.

Solution

$$3(\tan\theta + 1) = 2$$
$$\tan\theta + 1 = \frac{2}{3}$$
$$\tan\theta = \frac{2}{3} - 1$$

Rearrange the equation to isolate $\tan\theta$.

$$\tan\theta = -\frac{1}{3}$$

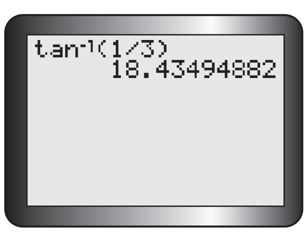


Since the tangent ratio is negative, x can be negative when y is positive, and vice versa.

The tangent ratio is negative in quadrants II and IV. The terminal arm of the angles lies in these two quadrants

There are two solutions for θ in the interval $0° \le \theta \le 360°$.

Determine the related acute angle using the inverse tangent function.



Evaluate $\tan^{-1}\left(\frac{1}{3}\right)$ using a calculator in degree mode, and round your answer to one decimal place.

$\tan^{-1}\left(\frac{1}{3}\right) \doteq 18.4°$, so the related acute angle is about 18.4°.

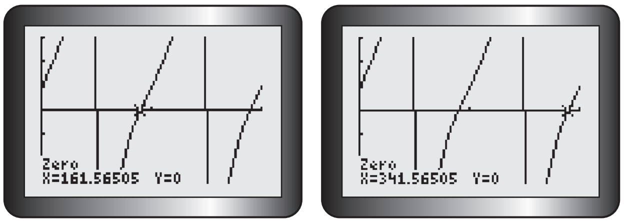
Subtract 18.4° from 180° to obtain the solution in quadrant II.
$\theta \doteq 180° - 18.4° = 161.6°$

If β is the related angle, the principal angle in quadrant II is $180° - \beta$. The principal angle in quadrant IV is $360° - \beta$.

Subtract 18.4° from 360° to obtain the solution in quadrant IV.
$\theta \doteq 360° - 18.4° = 341.6°$

θ is about 161.6° or 341.6°.

Verify the solutions by graphing $f(\theta) = 3(\tan \theta + 1) - 2$ in degree mode and determining the zeros in the given domain.



Choose window settings to match the domain $0 \leq \theta \leq 360°$.

The results confirm the solutions.

EXAMPLE 3 Solving a problem that involves a linear trigonometric equation

Today, the high tide in Matthews Cove, New Brunswick, is at midnight. The water level at high tide is 7.5 m. The depth, d metres, of the water in the cove at time t hours is modelled by the equation $d(t) = 4 + 3.5 \cos \frac{\pi}{6}t$. Jenny is planning a day trip to the cove tomorrow, but the water needs to be at least 2 m deep for her to manoeuvre her sailboat safely. How can Jenny determine the times when it will be safe for her to sail into Matthews Cove?

Solution

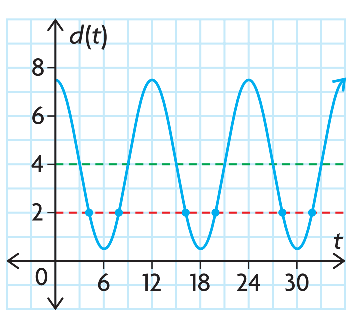
Draw a rough sketch of the depth function for at least the next 24 h, assuming that $t = 0$ is the high tide at midnight.

For the function $f(x) = a \cos kx + c$, the amplitude is a, the period is $\frac{2\pi}{k}$, and the horizontal axis is the line $y = c$. For the function $d(t) = 4 + 3.5 \cos \frac{\pi}{6}t$,

$a = 3.5$

$c = 4$

$$\text{period} = \frac{2\pi}{\frac{\pi}{6}} = 2\pi \times \frac{6}{\pi} = 12$$



From the graph, the water level will be near 2 m around 4 a.m., 8 a.m., 4 p.m., and 8 p.m.

It looks like the best time for her to enter the cove is around 8 a.m., and she needs to leave the cove around 4 p.m.

Determine the times when the water level is above 2 m and the times when the level equals 2 m.

$$4 + 3.5 \cos \frac{\pi}{6}t = 2$$

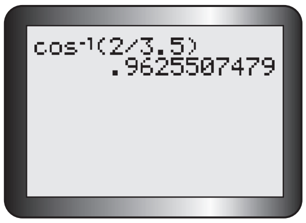
$$3.5 \cos \frac{\pi}{6}t = 2 - 4$$

$$\cos \frac{\pi}{6}t = \frac{-2}{3.5}$$

To get a better approximation of the times, solve the equation for $d(t) = 2$ to determine the related acute angle.

Since $4 + 3.5 \cos \frac{\pi}{6}t = 2$ is a linear trigonometric equation, isolate $\cos \frac{\pi}{6}t$.

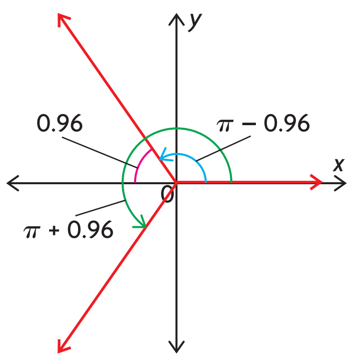
Determine the related acute angle.



Using a calculator in radian mode, determine the inverse cosine of $\frac{2}{3.5}$ to find the related acute angle.

$$\frac{\pi}{6}t \doteq 0.96$$

The related acute angle is about 0.96.



The cosine ratio is negative, so x is negative and r is positive. The terminal arms of $\frac{\pi}{6}t$ must lie in quadrants II and III.

To find the value of $\frac{\pi}{6}t$ in quadrant II, subtract the related acute angle from π.
$\pi - 0.96 = 2.18$

To find the value of $\frac{\pi}{6}t$ in quadrant III, add the related acute angle to π.
$\pi + 0.96 = 4.1$

The value of $\frac{\pi}{6}t$ is about 2.18 in quadrant II and about 4.1 in quadrant III.

To find the approximate times when the depth is 2 m, solve the following equations.

$$\frac{\pi}{6}t = 2.18 \quad \text{or} \quad \frac{\pi}{6}t = 4.1$$

Since Jenny is sailing tomorrow, the domain is $0 \le t \le 24$.

$$t = \frac{6}{\pi}(2.18) \qquad t = \frac{6}{\pi}(4.1)$$

$$t \doteq 4.16 \qquad t \doteq 7.83$$

$$t = 4.16 + 12 \qquad t = 7.83 + 12$$

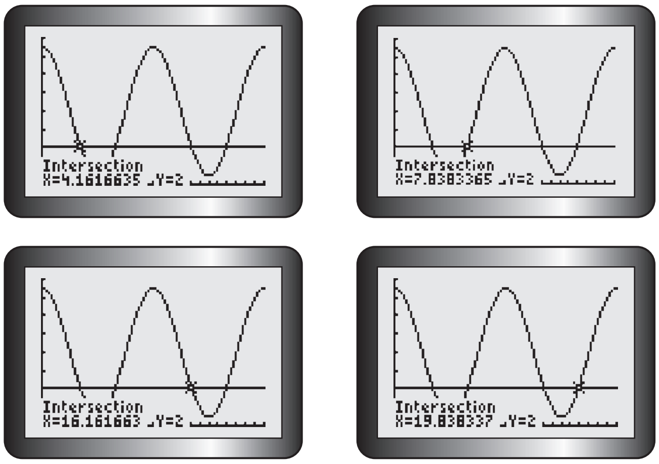
$$t = 16.16 \qquad t = 19.83$$

You can generate more solutions by adding 12, the period of the cosine function.

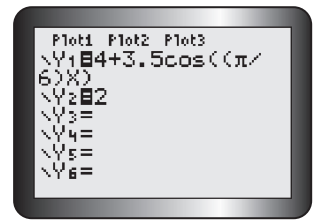
Jenny can safely sail into the cove when the water level is higher than 2 m. This occurs tomorrow, during the day, between 7:50 a.m. and 4:10 p.m.

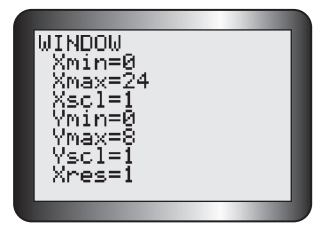
Multiply the digits to the right of the decimal by 60 to convert from a fraction of an hour to minutes. Tomorrow, the water level will be 2 m at about 4:10 a.m., 7:50 a.m., 4:10 p.m., and 7:50 p.m.

The water level is higher than 2 m when the tide function graph is above the line $d = 2$.



To verify the solution, graph $d(t)$ and the horizontal line $d = 2$ for the 24 h following midnight. Then determine the points of intersection.





The values of t are very close to the calculated values. Therefore, the solution is reasonable.

There is no need to convert the values of t into hours and minutes, since the values on the graph can be compared with the calculated solutions.

EXAMPLE 4 Selecting a strategy to solve a linear trigonometric equation that involves double angles

Solve $2 \sin \theta \cos \theta = \cos 2\theta$ for θ in the interval $0 \le \theta \le 2\pi$.

Solution

$$2 \sin \theta \cos \theta = \cos 2\theta$$
$$\sin 2\theta = \cos 2\theta$$

Use the $\sin 2\theta$ double angle formula to express the equation using the same argument.

$$\frac{\sin 2\theta}{\cos 2\theta} = \frac{\cos 2\theta}{\cos 2\theta}$$
$$\tan 2\theta = 1$$

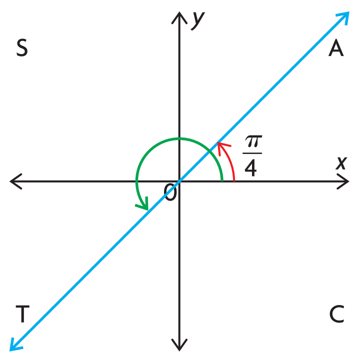
Divide both sides by $\cos 2\theta$ to express the equation using a single trigonometric function.

Solve $\tan 2\theta = 1$.
The related acute angle of 2θ is $\tan^{-1}(1) = \frac{\pi}{4}$.

Determine the related angle for 2θ by evaluating $\tan^{-1}(1)$.

Use the 1, 1, $\sqrt{2}$ special triangle to determine the inverse tangent of 1.

The tangent ratio is positive in quadrants I and III.



Since the tangent ratio is positive, x and y must have the same sign. This means that the terminal arm of 2θ lies in quadrant I or quadrant III.

The value of 2θ in quadrant I is $\frac{\pi}{4}$.

The value of 2θ in quadrant III is $\frac{5\pi}{4}$.

To find the value of 2θ in quadrant III, add the related angle to π.

$\pi + \frac{\pi}{4} = \frac{5\pi}{4}$.

To determine θ, solve the following equations.

$$2\theta = \frac{\pi}{4} \quad \text{or} \quad 2\theta = \frac{5\pi}{4}$$

$$\theta = \frac{\pi}{8} \qquad \theta = \frac{5\pi}{8}$$

$$\theta = \frac{\pi}{8} + \frac{\pi}{2} = \frac{5\pi}{8} \text{ (already determined)}$$

$$\theta = \frac{5\pi}{8} + \frac{\pi}{2} = \frac{9\pi}{8}$$

$$\theta = \frac{9\pi}{8} + \frac{\pi}{2} = \frac{13\pi}{8}$$

The period of $\tan 2\theta$ is $\frac{\pi}{2}$, so adding this to the two solutions will generate the other solutions in the given domain, $0 \leq \theta \leq 2\pi$.

Solutions for θ are $\frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}$, or $\frac{13\pi}{8}$.

In Summary

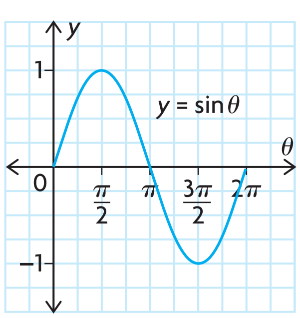
Key Idea

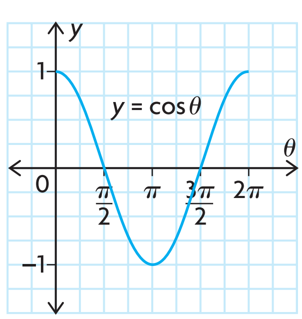
- The same strategies can be used to solve linear trigonometric equations when the variable is measured in degrees or radians.

Need to Know

- Because of their periodic nature, trigonometric equations have an infinite number of solutions. When we use a trigonometric model, we usually want solutions within a specified interval.
- To solve a linear trigonometric equation, use special triangles, a calculator, a sketch of the graph, and/or the CAST rule.
- A scientific or graphing calculator provides very accurate estimates of the value for an inverse trigonometric function. The inverse trigonometric function of a positive ratio yields the related angle. Use the related acute angle and the period of the corresponding function to determine all the solutions in the given interval.
- You can use a graphing calculator to verify the solutions for a linear trigonometric equation by
  - graphing the appropriate functions on the graphing calculator and determining the points of intersection
  - graphing an equivalent single function and determining its zeros

CHECK Your Understanding





1. Use the graph of $y = \sin \theta$ to estimate the value(s) of θ in the interval $0 \le \theta \le 2\pi$.

a) $\sin \theta = 1$
b) $\sin \theta = -1$
c) $\sin \theta = 0.5$
d) $\sin \theta = -0.5$
e) $\sin \theta = 0$
f) $\sin \theta = \frac{\sqrt{3}}{2}$

2. Use the graph of $y = \cos \theta$ to estimate the value(s) of θ in the interval $0 \le \theta \le 2\pi$.

a) $\cos \theta = 1$
b) $\cos \theta = -1$
c) $\cos \theta = 0.5$
d) $\cos \theta = -0.5$
e) $\cos \theta = 0$
f) $\cos \theta = \frac{\sqrt{3}}{2}$

3. Solve $\sin x = \frac{\sqrt{3}}{2}$, where $0 \le x \le 2\pi$.

a) How many solutions are possible?
b) In which quadrants would you find the solutions?
c) Determine the related acute angle for the equation.
d) Determine all the solutions for the equation.

4. Solve $\cos x = -0.8667$, where $0° \le x \le 360°$.

a) How many solutions are possible?
b) In which quadrants would you find the solutions?
c) Determine the related angle for the equation, to the nearest degree.
d) Determine all the solutions for the equation, to the nearest degree.

5. Solve $\tan \theta = 2.7553$, where $0 \le \theta \le 2\pi$.

a) How many solutions are possible?
b) In which quadrants would you find the solutions?
c) Determine the related angle for the equation, to the nearest hundredth.
d) Determine all the solutions for the equation, to the nearest hundredth.

PRACTISING

6. Determine the solutions for each equation, where $0 \le \theta \le 2\pi$.
K

a) $\tan \theta = 1$
b) $\sin \theta = \frac{1}{\sqrt{2}}$
c) $\cos \theta = \frac{\sqrt{3}}{2}$
d) $\sin \theta = -\frac{\sqrt{3}}{2}$
e) $\cos \theta = -\frac{1}{\sqrt{2}}$
f) $\tan \theta = \sqrt{3}$

7. Using a calculator, determine the solutions for each equation on the interval $0° \le \theta \le 360°$. Express your answers to one decimal place.

a) $2 \sin \theta = -1$
b) $3 \cos \theta = -2$
c) $2 \tan \theta = 3$
d) $-3 \sin \theta - 1 = 1$
e) $-5 \cos \theta + 3 = 2$
f) $8 - \tan \theta = 10$

8. Using a calculator, determine the solutions for each equation, to two decimal places, on the interval $0 \le x \le 2\pi$.

a) $3 \sin x = \sin x + 1$
b) $5 \cos x - \sqrt{3} = 3 \cos x$
c) $\cos x - 1 = -\cos x$
d) $5 \sin x + 1 = 3 \sin x$

9. Using a calculator, determine the solutions for each equation, to two decimal places, on the interval $0 \le x \le 2\pi$.

a) $2 - 2 \cot x = 0$
b) $\csc x - 2 = 0$
c) $7 \sec x = 7$
d) $2 \csc x + 17 = 15 + \csc x$
e) $2 \sec x + 1 = 6$
f) $8 + 4 \cot x = 10$

10. Using a calculator, determine the solutions for each equation, to two decimal places, on the interval $0 \le x \le 2\pi$.

a) $\sin 2x = \frac{1}{\sqrt{2}}$
b) $\sin 4x = \frac{1}{2}$
c) $\sin 3x = -\frac{\sqrt{3}}{2}$
d) $\cos 4x = -\frac{1}{\sqrt{2}}$
e) $\cos 2x = -\frac{1}{2}$
f) $\cos \frac{x}{2} = \frac{\sqrt{3}}{2}$

11. A city's daily high temperature, in degrees Celsius, can be modelled by the function $t(d) = -28 \cos \frac{2\pi}{365}d + 10$, where d is the day of the year and $1 =$ January 1. On days when the temperature is approximately 32 °C or above, the air conditioners at city hall are turned on. During what days of the year are the air conditioners running at city hall?

A

12. The height, in metres, of a nail in a water wheel above the surface of the water, as a function of time, can be modelled by the function $h(t) = -4 \sin \frac{\pi}{4}(t - 1) + 2.5$, where t is the time in seconds. During what periods of time is the nail below the water in the first 24 s that the wheel is rotating?

13. Solve $\sin\left(x + \frac{\pi}{4}\right) = \sqrt{2} \cos x$ for $0 \leq x \leq 2\pi$.

T

14. Sketch the graph of $y = \sin 2\theta$ for $0 \leq \theta \leq 2\pi$. On the graph, clearly indicate all the solutions for the trigonometric equation $\sin 2\theta = -\frac{1}{\sqrt{2}}$.

C

15. Explain why the value of the function $f(x) = 25 \sin \frac{\pi}{50}(x + 20) - 55$ at $x = 3$ is the same as the value of the function at $x = 7$.

16. Create a table like the one below to compare the algebraic and graphical strategies for solving a trigonometric equation. In what ways are the strategies similar, and in what ways are they different? Use examples in your comparison.

	Method for Solving	
	Algebraic Strategy	**Graphical Strategy**
Similarities		
Differences		

Extending

17. Solve the trigonometric equation $2 \sin x \cos x + \sin x = 0$. (*Hint:* You may find it helpful to factor the left side of the equation.)

18. Solve each equation for $0 \leq x \leq 2\pi$.

a) $\sin 2x - 2 \cos^2 x = 0$ **b)** $3 \sin x + \cos 2x = 2$