7.6 Solving Quadratic Trigonometric Equations

GOAL

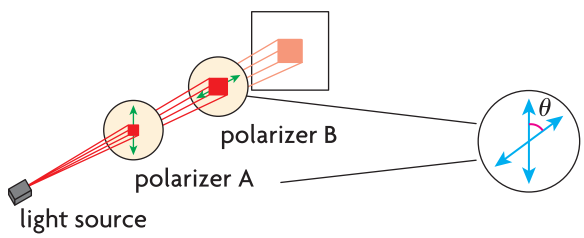
Solve quadratic trigonometric equations using graphs and algebra.

YOU WILL NEED

- graphing calculator

LEARN ABOUT the Math

A polarizing material is used in camera lens filters, LCD televisions, and sunglasses to reduce glare. In these examples, two polarizers are used to reduce the intensity of the light that enters your eyes.



The amount of the reduction in light intensity, I, depends on θ, the acute angle formed between the axis of polarizer A and the axis of polarizer B. Malus's law states that $I = I_0 \cos^2 \theta$, where I_0 is the intensity of the initial beam of light and I is the intensity of the light emerging from the polarizing material.

? At what angle to the axis of polarizer A should polarizer B be placed to reduce the light intensity by 97%?

EXAMPLE 1 Solving a quadratic trigonometric equation using an algebraic strategy

Use Malus's law to determine the angle between polarizer A and polarizer B that will reduce the light intensity by 97%.

Solution

Malus's law is $I = I_0 \cos^2 \theta$.
Solve the equation $0.03\, I_0 = I_0 \cos^2 \theta$.

If the light intensity is reduced by 97%, then it is $1 - 0.97$ or 0.03 of the initial intensity. Therefore, $I = 0.03 I_0$.

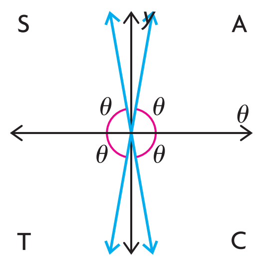
$$\frac{0.03I_0}{I_0} = \frac{I_0 \cos^2 \theta}{I_0}$$

Divide both sides by I_0 to isolate $\cos \theta$.

$$0.03 = \cos^2 \theta$$
$$\pm\sqrt{0.03} = \sqrt{\cos^2 \theta}$$
$$\pm 0.1732 = \cos \theta$$

Take the square root of both sides.

$\cos \theta = 0.1732$ or $\cos \theta = -0.1732$



Since the cosine ratio has both positive and negative values, solving both equations will result in values for θ that lie in all four quadrants.

This means that there are four possible solutions. These solutions, however, are all related by the acute related angle.

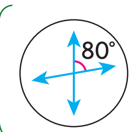
Only the acute angle is necessary. This is the angle in quadrant I.

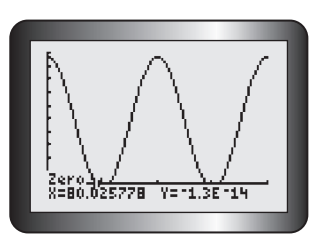
$$\cos \theta = 0.1732$$
$$\theta = \cos^{-1}(0.1732)$$

To determine the related acute angle, use a calculator in degree mode and determine the inverse cosine of 0.1732.

$$\theta \doteq 80°$$

To reduce the light intensity by 97%, the axis of polarizing material B must be placed at an angle of about 80° to the axis of polarizing material A.





To verify the solution, graph $f(x) = \cos^2 \theta - 0.03$ in degree mode and determine its first zero.

The graph confirms the calculated solution.

Reflecting

A. Compare the number of solutions between 0° and 360° for the equation $\cos^2 x = 0.03$ with the number of solutions for a linear trigonometric equation, such as $\cos x = 0.03$. Explain the difference, using both graphical and algebraic analyses.

B. Why were some of the solutions for the trigonometric equation $\cos^2 x = 0.03$ omitted in the context of Example 1?

C. How would the equation change if the intensity of light in an LCD television was reduced by 25%? What angle would be needed between the axis of polarizer A and the axis of polarizer B for this situation?

APPLY the Math

EXAMPLE 2 Selecting a factoring strategy to solve quadratic trigonometric equations

Solve each equation for x in the interval $0 \leq x \leq 2\pi$. Verify your solutions by graphing.

a) $\sin^2 x - \sin x = 2$ **b)** $2\sin^2 x - 3\sin x + 1 = 0$

Solution

a)
$$\sin^2 x - \sin x = 2$$
$$\sin^2 x - \sin x - 2 = 0$$
$$(\sin x - 2)(\sin x + 1) = 0$$
$\sin x = 2$ or $\sin x = -1$

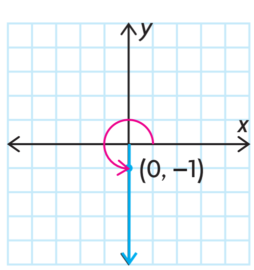
Subtract 2, so you have 0 on the right side. This is a quadratic equation in $\sin x$. Factor.

Solve both of these equations.

The equation $\sin x = 2$ has no solutions.

Since the graph of $y = \sin x$ has the range $\{y \in \mathbf{R} \mid -1 \leq y \leq 1\}$, the values of $\sin x$ cannot exceed 1.

The equation $\sin x = -1$ has only one solution in the interval $0 \leq x \leq 2\pi$.

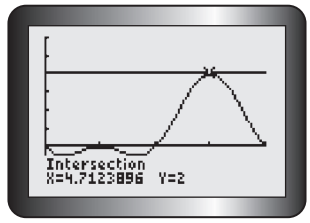


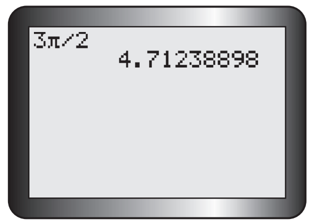
Since $\sin x = \frac{y}{r} = \frac{-1}{1}$, the point $(0, -1)$ lies on the terminal arm of angle x.

The solution is $x = \frac{3\pi}{2}$.

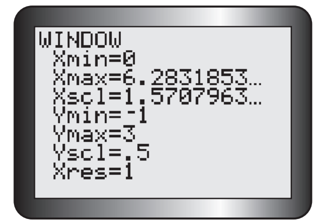
Tech Support

For help using the graphing calculator to determine points of intersection, see Technical Appendix, T-12.





To verify the solution, graph $f(x) = \sin^2 x - \sin x$ and $g(x) = 2$ in the required interval. Then determine the points of intersection.



You can see that there is only one solution in the interval $0 \le x \le 2\pi$.

Since $\frac{3\pi}{2} \doteq 4.712\ 388\ 98$, this verifies the previous solution.

b)

$$2\sin^2 x - 3\sin x + 1 = 0$$

$$(2\sin x - 1)(\sin x - 1) = 0$$

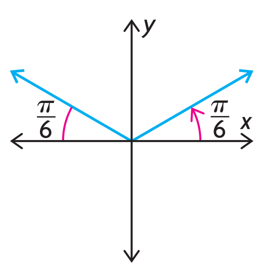
Factor the left side.

$$\sin x = \frac{1}{2} \text{ or } \sin x = 1$$

$\sin x = \frac{1}{2}$ has two solutions in $0 \le x \le 2\pi$.

Use the 1, 2, $\sqrt{3}$ special triangle to determine that $\sin\frac{\pi}{6} = \frac{1}{2}$.

$\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$ is the solution in quadrant I and is also the related acute angle.



Since $\sin x$ is positive, both y and r are positive. The solutions lie in quadrants I and II.

To determine the solution in quadrant II, subtract the related angle from π.

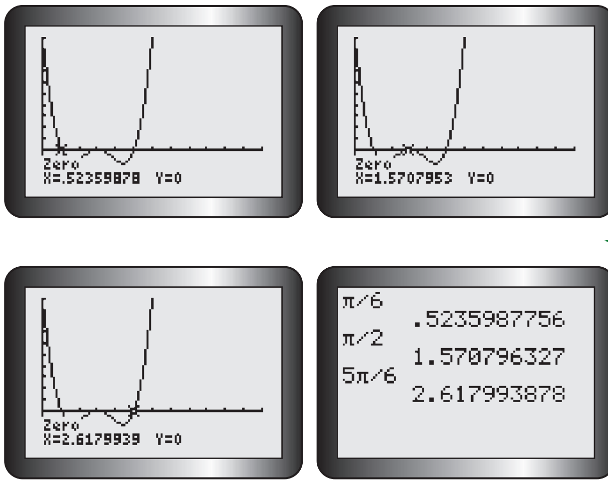
$$\pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

The solution in quadrant II is $\frac{5\pi}{6}$.

$\sin x = 1$ has one solution.
This occurs when $x = \frac{\pi}{2}$.

$$\sin^{-1}(1) = \frac{\pi}{2}$$

Graph $f(x) = 2\sin^2 x - 3\sin x + 1$, and determine the zeros to verify the solutions.



Limit the window to the interval $[0, 2\pi]$ so you only consider the required solutions.

If you set Xscl to $\frac{\pi}{6}$, you can see that the zeros match the solutions already obtained.

The solutions match those obtained algebraically.

EXAMPLE 3 Selecting a strategy using identities to solve quadratic trigonometric equations

For each equation, use a trigonometric identity to create a quadratic equation. Then solve the equation for x in the interval $[0, 2\pi]$.

a) $2\sec^2 x - 3 + \tan x = 0$ **b)** $3\sin x + 3\cos 2x = 2$

Solution

a)
$$2\sec^2 x - 3 + \tan x = 0$$
$$2(1 + \tan^2 x) - 3 + \tan x = 0$$

Use the Pythagorean identity $1 + \tan^2 x = \sec^2 x$ to create an equation with only $\tan x$ and $\tan^2 x$ in it.

$$2 + 2\tan^2 x - 3 + \tan x = 0$$
$$2\tan^2 x + \tan x - 1 = 0$$

Expand and combine terms. Factor.

$$(2\tan x - 1)(\tan x + 1) = 0$$
$$2\tan x - 1 = 0 \text{ or } \tan x + 1 = 0$$

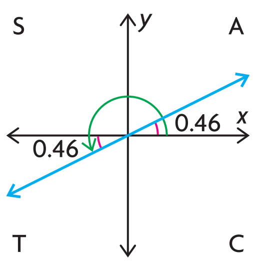
Set each factor equal to 0 to solve the equations.

$$\tan x = \frac{1}{2} \quad \text{or} \quad \tan x = -1$$

$\tan x = \frac{1}{2}$ has solutions in quadrants I and III.

$\tan^{-1}\left(\frac{1}{2}\right) \doteq 0.46$

This is the solution in quadrant I and is also the related angle.

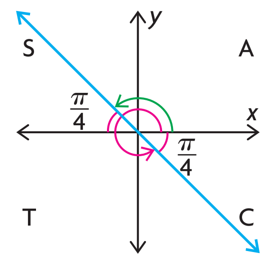


The solution in quadrant III is $\pi + 0.46 \doteq 3.60$

$\tan x = -1$ has solutions in quadrants II and IV.

$\tan^{-1}(1) = \frac{\pi}{4}$

The related angle is $\frac{\pi}{4}$.



The solution in quadrant II is $\pi - \frac{\pi}{4} = \frac{3\pi}{4}$.

The solution in quadrant IV is $2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$.

Use the CAST rule to help determine the solutions in the required interval, $0 \leq x \leq 2\pi$.

Solutions to the equation are $x \doteq 0.46, \frac{3\pi}{4}, 3.60$, or $\frac{7\pi}{4}$ radians, rounded to two decimal places where not exact.

Round answers that are not exact.

b)

$$\begin{aligned}
3 \sin x + 3 \cos 2x &= 2 \\
3 \sin x + 3(1 - 2 \sin^2 x) &= 2 \\
3 \sin x + 3 - 6 \sin^2 x &= 2 \\
0 &= 2 - 3 \sin x - 3 + 6 \sin^2 x \\
0 &= 6 \sin^2 x - 3 \sin x - 1
\end{aligned}$$

To create a single trigonometric function (such as sin x) with the same argument, use the double angle formula $\cos 2x = 1 - 2 \sin^2 x$. Rearrange the equation so that one side equals 0.

$$0 = 6a^2 - 3a - 1$$

$$a = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(6)(-1)}}{2(6)}$$

$$a = \frac{3 \pm \sqrt{33}}{12}$$

$a \doteq 0.73$ or $a \doteq -0.23$

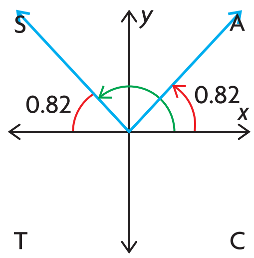
$\sin x = 0.73$ or $\sin x = -0.23$

This is not factorable, so substitute $a = \sin x$ and use the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

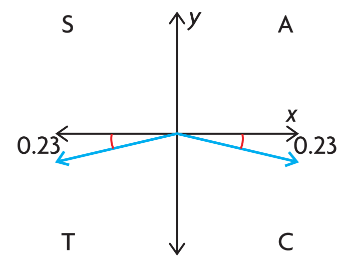
where $a = 6$, $b = -3$, and $c = -1$.

sin $x = 0.73$ has solutions in quadrants I and II.
$\sin^{-1}(0.73) \doteq 0.82$
This is the solution in quadrant I and is also the related angle.



The other solution is $\pi - 0.82 = 2.32$.

sin $x = -0.23$ has solutions in quadrants III and IV.
$\sin^{-1}(0.23) \doteq 0.23$. The related angle is 0.23.



Use the CAST rule to help determine the solutions in the required interval, $0 \leq x \leq 2\pi$.

The solution in quadrant III is $\pi + 0.23 = 3.37$.
The solution in quadrant IV is $2\pi - 0.23 = 6.05$.

The solutions are approximately 0.82, 2.32, 3.37, or 6.05.

In Summary

Key Ideas

- In some applications, the formula contains a square of a trigonometric ratio. This leads to a quadratic trigonometric equation that can be solved algebraically or graphically.
- A quadratic trigonometric equation may have multiple solutions in the interval $0 \leq x \leq 2\pi$. Some of the solutions may be inadmissible, however, in the context of the problem.

Need to Know

- You can often factor a quadratic trigonometric equation and then solve the resulting two linear trigonometric equations. In cases where the equation cannot be factored, use the quadratic formula and then solve the resulting linear trigonometric equations.

Note: The solutions to $ax^2 + bx + c = 0$ are determined by $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

- You may need to use a Pythagorean identity, compound angle formula, or double angle formula to create a quadratic equation that contains only a single trigonometric function whose arguments all match.

CHECK Your Understanding

1. Factor each expression.

a) $\sin^2\theta - \sin\theta$
b) $\cos^2\theta - 2\cos\theta + 1$
c) $3\sin^2\theta - \sin\theta - 2$
d) $4\cos^2\theta - 1$
e) $24\sin^2 x - 2\sin x - 2$
f) $49\tan^2 x - 64$

2. Solve the first equation in each pair of equations for y and/or z. Then use the same strategy to solve the second equation for x in the interval $0 \le x \le 2\pi$.

a) $y^2 = \frac{1}{3}$, $\tan^2 x = \frac{1}{3}$

b) $y^2 + y = 0$, $\sin^2 x + \sin x = 0$

c) $y - 2yz = 0$, $\cos x - 2 \cos x \sin x = 0$

d) $yz = y$, $\tan x \sec x = \tan x$

3. a) Solve the equation $6y^2 - y - 1 = 0$.

b) Solve $6 \cos^2 x - \cos x - 1 = 0$ for $0 \le x \le 2\pi$.

PRACTISING

4. Solve for θ, to the nearest degree, in the interval $0° \le \theta \le 360°$.

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a) $\sin^2 \theta = 1$

b) $\cos^2 \theta = 1$

c) $\tan^2 \theta = 1$

d) $4 \cos^2 \theta = 1$

e) $3 \tan^2 \theta = 1$

f) $2 \sin^2 \theta = 1$

5. Solve each equation for x, where $0° \le x \le 360°$.

a) $\sin x \cos x = 0$

b) $\sin x (\cos x - 1) = 0$

c) $(\sin x + 1) \cos x = 0$

d) $\cos x (2 \sin x - \sqrt{3}) = 0$

e) $(\sqrt{2} \sin x - 1)(\sqrt{2} \sin x + 1) = 0$

f) $(\sin x - 1)(\cos x + 1) = 0$

6. Solve each equation for x, where $0 \le x \le 2\pi$.

a) $(2 \sin x - 1) \cos x = 0$

b) $(\sin x + 1)^2 = 0$

c) $(2 \cos x + \sqrt{3}) \sin x = 0$

d) $(2 \cos x - 1)(2 \sin x + \sqrt{3}) = 0$

e) $(\sqrt{2} \cos x - 1)(\sqrt{2} \cos x + 1) = 0$

f) $(\sin x + 1)(\cos x - 1) = 0$

7. Solve for θ to the nearest hundredth, where $0 \le \theta \le 2\pi$.

a) $2 \cos^2 \theta + \cos \theta - 1 = 0$

b) $2 \sin^2 \theta = 1 - \sin \theta$

c) $\cos^2 \theta = 2 + \cos \theta$

d) $2 \sin^2 \theta + 5 \sin \theta - 3 = 0$

e) $3 \tan^2 \theta - 2 \tan \theta = 1$

f) $12 \sin^2 \theta + \sin \theta - 6 = 0$

8. Solve each equation for x, where $0 \le x \le 2\pi$.

a) $\sec x \csc x - 2 \csc x = 0$

b) $3 \sec^2 x - 4 = 0$

c) $2 \sin x \sec x - 2\sqrt{3} \sin x = 0$

d) $2 \cot x + \sec^2 x = 0$

e) $\cot x \csc^2 x = 2 \cot x$

f) $3 \tan^3 x - \tan x = 0$

9. Solve each equation in the interval $0 \leq x \leq 2\pi$. Round to two decimal places, if necessary.

a) $5 \cos 2x - \cos x + 3 = 0$

b) $10 \cos 2x - 8 \cos x + 1 = 0$

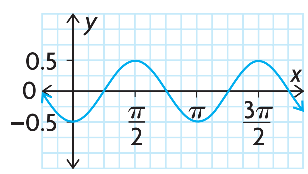
c) $4 \cos 2x + 10 \sin x - 7 = 0$

d) $-2 \cos 2x = 2 \sin x$

10. Solve the equation $8 \sin^2 x - 8 \sin x + 1 = 0$ in the interval $0 \leq x \leq 2\pi$.

11. The quadratic trigonometric equation $\cot^2 x - b \cot x + c = 0$ has the solutions $\frac{\pi}{6}, \frac{\pi}{4}, \frac{7\pi}{6}$, and $\frac{5\pi}{4}$ in the interval $0 \leq x \leq 2\pi$. What are the values of b and c?

12. The graph of the quadratic trigonometric equation $\sin^2 x - c = 0$ is shown. What is the value of c?



13. **A** Natasha is a marathon runner, and she likes to train on a 2π km stretch of rolling hills. The height, in kilometres, of the hills above sea level, relative to her home, can be modelled by the function $h(d) = 4 \cos^2 d - 1$, where d is the distance travelled in kilometres. At what intervals in the stretch of rolling hills is the height above sea level, relative to Natasha's home, less than zero?

14. **T** Solve the equation $6 \sin^2 x = 17 \cos x + 11$ for x in the interval $0 \leq x \leq 2\pi$.

15. **a)** Solve the equation $\sin^2 x - \sqrt{2} \cos x = \cos^2 x + \sqrt{2} \cos x + 2$ for x in the interval $0 \leq x \leq 2\pi$.

b) Write a general solution for the equation in part a).

16. **C** Explain why it is possible to have different numbers of solutions for quadratic trigonometric equations. Give examples to illustrate your explanation.

Extending

17. Given that $f(x) = \frac{\tan x}{1 - \tan x} - \frac{\cot x}{1 - \cot x}$, determine all the values of a in the interval $0 \leq a \leq 2\pi$, such that $f(x) = \tan (x + a)$.

18. Solve the equation $2 \cos 3x + \cos 2x + 1 = 0$.

19. Solve $3 \tan^2 2x = 1$, $0° \leq x \leq 360°$.

20. Solve $\sqrt{2} \sin \theta = \sqrt{3} - \cos \theta$, $0 \leq \theta \leq 2\pi$.