8.1 Exploring the Logarithmic Function

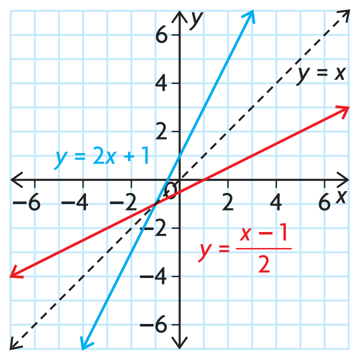
YOU WILL NEED

- graph paper

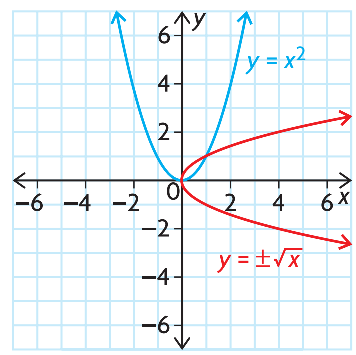
GOAL

Investigate the inverse of the exponential function.

EXPLORE the Math



The inverse of a linear function, such as $f(x) = 2x + 1$, is linear.



The inverse of a quadratic function, such as $g(x) = x^2$, has a shape that is congruent to the shape of the original function.

? What does the graph of the inverse of an exponential function like $y = 2^x$ look like, and what are its characteristics?

A. Consider the function $h(x) = 2^x$. Create a table of values, using integer values for the domain $-3 \leq x \leq 4$.

B. On graph paper, graph the exponential function in part A. State the domain and range of this function.

C. Interchange x and y in the equation for h to obtain the equation of the inverse relation. Create a table of values for this inverse relation. How does each y-value of this relation relate to the base, 2, and its corresponding x-value?

D. On the same axes that you used to graph the exponential function in part B, graph the inverse. Is the inverse a function? Explain.

E. Graph the line $y = x$ on the same axes. How do the graphs of the exponential function $h(x) = 2^x$ and the graph of the logarithmic function $h^{-1}(x) = \log_2 x$ relate to this line?

logarithmic function

The inverse of the exponential function $y = a^x$ is the function with exponential equation $x = a^y$. We write y as a function of x using the logarithmic form of this equation, $y = \log_a x$. As with the exponential function, $a > 0$ and $a \neq 1$.

F. Repeat parts A to E, first using $j(x) = 10^x$ and then using $k(x) = \left(\frac{1}{2}\right)^x$.

G. State the domain and range of the inverses of $h(x)$, $j(x)$, and $k(x)$.

H. How is the range of each logarithmic function related to the domain of its corresponding exponential function? How is the domain of the logarithmic function related to the range of the corresponding exponential function?

I. How would you describe these logarithmic functions? Create a summary table that includes information about intercepts, asymptotes, and shapes of the graphs.

Reflecting

J. What point is common to the graphs of all three logarithmic functions?

K. How are the graphs of an exponential function and the logarithmic function with the same base related?

L. How are the graphs of $h(x) = 2^x$ and $k(x) = \left(\frac{1}{2}\right)^x$ related? How are the graphs of $h^{-1}(x) = \log_2 x$ and $k^{-1}(x) = \log_{\frac{1}{2}} x$ related?

M. How does the value of a in $y = a^x$ influence the graph of $y = \log_a x$? How might you have predicted this?

N. The graph of $h^{-1}(x) = \log_2 x$ includes the point $(8, 3)$. Therefore, $3 = \log_2 8$. What is the value of $\log_2 16$ What meaning does $\log_2 x$ have? More generally, what meaning does the expression $\log_a x$ have?

In Summary

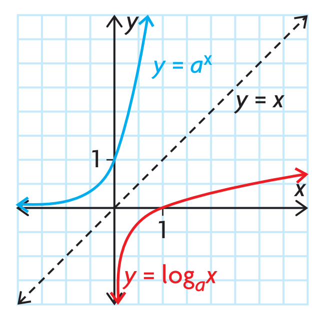
Key Ideas

- The inverse of the exponential function $y = a^x$ is also a function. It can be written as $x = a^y$. (This is the exponential form of the inverse.) An equivalent form of $x = a^y$ is $y = \log_a x$. (This is the logarithmic form of the inverse and is read as "the **logarithm** of x to the base a.") The function $y = \log_a x$ is called the logarithmic function.
- Since $x = a^y$ and $y = \log_a x$ are equivalent, a logarithm is an exponent. The expression $\log_a x$ means "the exponent that must be applied to base $\boldsymbol{a}$ to get the value of x." For example, $\log_2 8 = 3$ since $2^3 = 8$.

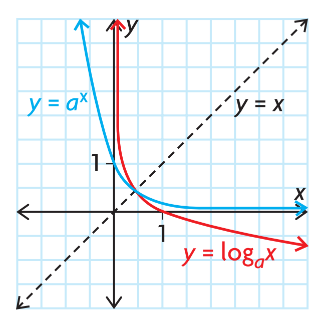
Need to Know

- The general shape of the graph of the logarithmic function depends on the value of the base.

When $a > 1$, the exponential function is an increasing function, and the logarithmic function is also an increasing function.



When $0 < a < 1$, the exponential function is a decreasing function and the logarithmic function is also a decreasing function.



- The y-axis is the vertical asymptote for the logarithmic function. The x-axis is the horizontal asymptote for the exponential function.
- The x-intercept of the logarithmic function is 1, while the y-intercept of the exponential function is 1.
- The domain of the logarithmic function is $\{x \in \mathbf{R} \mid x > 0\}$, since the range of the exponential function is $\{y \in \mathbf{R} \mid y > 0\}$.
- The range of the logarithmic function is $\{y \in \mathbf{R}\}$, since the domain of the exponential function is $\{x \in \mathbf{R}\}$.

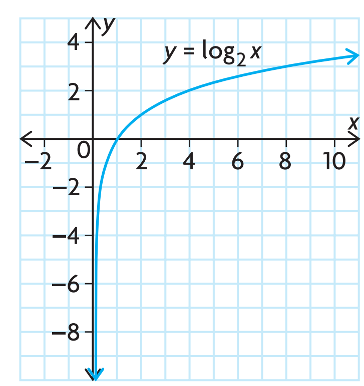
FURTHER Your Understanding

1. Sketch a graph of the inverse of each exponential function.
   a) $f(x) = 4^x$
   b) $f(x) = 8^x$
   c) $f(x) = \left(\frac{1}{3}\right)^x$
   d) $f(x) = \left(\frac{1}{5}\right)^x$

2. Write the equation of each inverse function in question 1 in
   i) exponential form
   ii) logarithmic form

3. Compare the key features of the graphs in question 1.

4. Explain how you can use the graph of $y = \log_2 x$ (at right) to help you determine the solution to $2^y = 8$.



5. Write the equation of the inverse of each exponential function in exponential form.
   a) $y = 3^x$
   b) $y = 10^x$
   c) $y = \left(\frac{1}{4}\right)^x$
   d) $y = m^x$

6. Write the equation of the inverse of each exponential function in question 5 in logarithmic form.

7. Write the equation of each of the following logarithmic functions in exponential form.
   a) $y = \log_5 x$
   b) $y = \log_{10} x$
   c) $y = \log_3 x$
   d) $y = \log_{\frac{1}{4}} x$

8. Write the equation of the inverse of each logarithmic function in question 7 in exponential form.

9. Evaluate each of the following:
   a) $\log_2 4$
   b) $\log_3 27$
   c) $\log_4 64$
   d) $\log_5 1$
   e) $\log_2\left(\frac{1}{2}\right)$
   f) $\log_3 \sqrt{3}$

10. Why can $\log_3(-9)$ not be evaluated?

11. For each of the following logarithmic functions, write the coordinates of the five points that have y-values of $-2, -1, 0, 1, 2$.
   a) $y = \log_2 x$
   b) $y = \log_{10} x$