8.2 Transformations of Logarithmic Functions

YOU WILL NEED

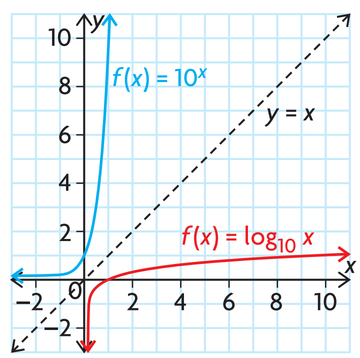
- graphing calculator

GOAL

Determine the effects of varying the parameters of the graph of $y = a \log_{10}(k(x - d)) + c$.

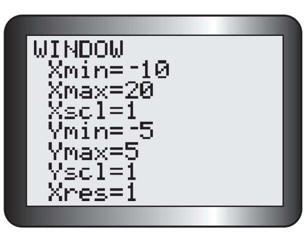
INVESTIGATE the Math

The function $f(x) = \log_{10}x$ is an example of a logarithmic function. It is the inverse of the exponential function $f(x) = 10^x$.



? How does varying the parameters of a function in the form $g(x) = a \log_{10}(k(x - d)) + c$ affect the graph of the parent function, $f(x) = \log_{10}x$?

A. The log button on a graphing calculator represents $\log_{10}x$. Graph $y = \log_{10}x$ on a graphing calculator. Use the window setting shown.



B. Consider the following functions:

- $y = \log_{10}(x - 2)$
- $y = \log_{10}(x - 4)$
- $y = \log_{10}(x + 4)$

Make a conjecture about the type of transformation that must be applied to the graph of $y = \log_{10}x$ to graph each of these functions.

C. Graph the functions in part B along with the graph of $y = \log_{10}x$. Compare each of these graphs with the graph of $y = \log_{10}x$. Was your conjecture correct? Summarize the transformations that are applied to $y = \log_{10}x$ to obtain $y = \log_{10}(x - d)$.

Communication | *Tip*

If there is no value of a in a logarithmic function ($\log_a x$), the base is understood to be 10; that is, $\log x = \log_{10}x$. Logarithms with base 10 are called common logarithms.

D. Examine the following functions:

- $y = \log_{10}x + 3$
- $y = \log_{10}x - 4$

Make a conjecture about the type of transformation that must be applied to the graph of $y = \log_{10}x$ to graph each of these functions.

E. Delete all but the first function in the equation editor, and enter the functions in part D. Graph the functions. Compare each of these graphs with the graph of $y = \log_{10}x$. Was your conjecture correct? Summarize the transformations that are applied to $y = \log_{10}x$ to obtain $y = \log_{10}x + c$.

F. State the transformations that you would need to apply to $y = \log_{10}x$ to graph the function $y = \log_{10}(x - d) + c$.

G. Make a conjecture about the transformations that you would need to apply to $y = \log_{10}x$ to graph each of the following functions:

- $y = 2 \log_{10}x$
- $y = \frac{1}{3} \log_{10}x$
- $y = -2 \log_{10}x$

H. Delete all but the first function in the equation editor, and enter the functions in part G. Graph the functions. Compare each of these graphs with the graph of $y = \log_{10}x$. Was your conjecture correct? Summarize the transformations that are applied to $y = \log_{10}x$ to obtain $y = a \log_{10}x$.

I. Make a conjecture about the transformations that you would need to apply to $y = \log_{10}x$ to graph each of the following functions:

- $y = \log_{10}(2x)$
- $y = \log_{10}\left(\frac{1}{5}x\right)$
- $y = \log_{10}(-2x)$

J. Delete all but the first function in the equation editor, and enter the functions in part I. Graph the functions. Compare each of these graphs with the graph of $y = \log_{10}x$. Was your conjecture correct? Summarize the transformations that are applied to $y = \log_{10}x$ to obtain $y = \log_{10}(kx)$.

K. What transformations must be applied to $y = \log_{10}x$ to graph $y = a \log_{10}(kx)$?

Reflecting

L. Describe the domain and range of $y = \log_{10}(x - d)$, $y = \log_{10}x + c$, $y = \log_{10}(kx)$, and $y = a\log_{10}x$.

M. How do the algebraic representations of the functions resulting from transformations of logarithmic functions compare with the algebraic representations of the functions resulting from transformations of polynomial, trigonometric, and exponential functions?

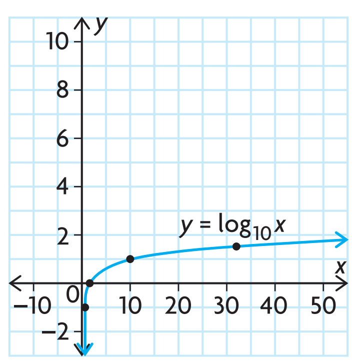
N. Identify the transformations that are related to the parameters *a, k, d,* and *c* in the general logarithmic function $y = a\,(\log_{10}k(x - d)) + c$.

APPLY the Math

EXAMPLE 1 Connecting transformations of a logarithmic function to key points of $y = \log_{10}x$

Use transformations to sketch the function $y = -2\log_{10}(x - 4)$. State the domain and range.

Solution



Sketch $y = \log_{10}x$. Choose some points on the graph, such as $\left(\frac{1}{10}, -1\right)$, (1, 0), (10, 1), and the estimated point (32, 1.5). Use these points as key points to help graph the transformed function. The vertical asymptote is the *y*-axis, $x = 0$. Apply transformations in the same order used for all functions: stretches/compressions/reflections first, followed by translations.

$(x, y) \rightarrow (x, -2y)$

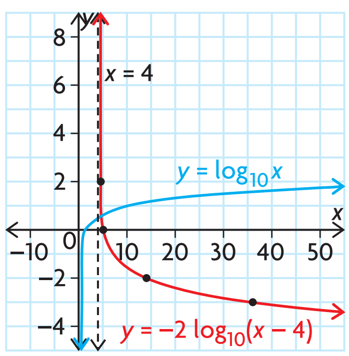
Parent Function $y = \log_{10}x$	Stretched/Reflected Function $y = -2\log_{10}x$
$\left(\frac{1}{10}, -1\right)$	$\left(\frac{1}{10}, -2(-1)\right) = \left(\frac{1}{10}, 2\right)$
$(1, 0)$	$(1, -2(0)) = (1, 0)$
$(10, 1)$	$(10, -2(1)) = (10, -2)$
$(32, 1.5)$	$(32, -2(1.5)) = (32, -3)$

The parent function is changed by multiplying all the *y*-coordinates by -2, resulting in a vertical stretch of factor 2 and a reflection in the *x*-axis.

$(x, -2y) \rightarrow (x + 4, -2y)$

Stretched/Reflected Function $y = -2 \log_{10} x$	**Final Transformed Function $y = -2 \log_{10}(x - 4)$**
$\left(\frac{1}{10}, 2\right)$	$\left(\frac{1}{10} + 4, 2\right) = \left(4\frac{1}{10}, 2\right)$
$(1, 0)$	$(1 + 4, 0) = (5, 0)$
$(10, -2)$	$(10 + 4, -2) = (14, -2)$
$(32, -3)$	$(32 + 4, -3) = (36, -3)$

Adding 4 to the *x*-coordinate of each of the transformed points results in a horizontal translation 4 units to the right.



Plot the new points and draw the graph.
The vertical asymptote is now $x = 4$ because of the translation to the right.

Domain $= \{x \in \mathbf{R} \mid x > 4\}$

The values of *x* must all be greater than 4 since the curve is to the right of the vertical asymptote.

Range $= \{y \in \mathbf{R}\}$

The range of the original function was not changed by the transformations.

EXAMPLE 2 Connecting a geometric description of a function to an algebraic representation

The logarithmic function $y = \log_{10}x$ has been vertically compressed by a factor of $\frac{2}{3}$, horizontally stretched by a factor of 4, and then reflected in the y-axis. It has also been horizontally translated so that the vertical asymptote is $x = -2$ and then vertically translated 3 units down. Write an equation of the transformed function, and state its domain and range.

Solution

$$y = a\log_{10}(k(x - d)) + c$$

Write the general form of the logarithmic equation.

$$y = \frac{2}{3}\log_{10}\left(-\frac{1}{4}(x + 2)\right) - 3$$

Since the function has been vertically compressed by a factor of $\frac{2}{3}$, $a = \frac{2}{3}$.

Since the function has been horizontally stretched by a factor of 4, $\frac{1}{k} = 4$, so $k = \frac{1}{4}$.

The function has been reflected in the y-axis, so k is negative.

The vertical asymptote of the parent function is $x = 0$.

Since the asymptote of the transformed function is $x = -2$, the parent function has been horizontally translated 2 units left, so $d = -2$.

The function has been vertically translated 3 units down, so $c = -3$.

$$\text{Domain} = \{x \in \mathbf{R} \mid x < -2\}$$

The curve is to the left of the vertical asymptote, so the domain is $x < -2$.

$$\text{Range} = \{y \in \mathbf{R}\}$$

The range is the same as the range of the parent function.

In Summary

Key Ideas

- A logarithmic function of the form $f(x) = a\log_{10}(k(x - d)) + c$ can be graphed by applying the appropriate transformations to the parent function, $f(x) = \log_{10}x$.
- To graph a transformed logarithmic function, apply the stretches/compressions/reflections given by parameters a and k first. Then apply the vertical and horizontal translation given by the parameters c and d.

Need to Know

- Consider a logarithmic function of the form $f(x) = a\log_{10}(k(x - d)) + c$.

Transformations of the Parent Function
$\|a\|$ gives the vertical stretch/compression factor. If $a < 0$, there is also a reflection in the x-axis.
$\left\|\frac{1}{k}\right\|$ gives the horizontal stretch/compression factor. If $k < 0$, there is also a reflection in the y-axis.
d gives the horizontal translation.
c gives the vertical translation.

- The vertical asymptote changes when a horizontal translation is applied. The domain of a transformed logarithmic function depends on where the vertical asymptote is located and whether the function is to the left or the right of the vertical asymptote. If the function is to the left of the asymptote $x = d$, the domain is $x < d$. If it is to the right of the asymptote, the domain is $x > d$.
- The range of a transformed logarithmic function is always $\{y \in \mathbf{R}\}$.

CHECK Your Understanding

1. Each of the following functions is a transformation of $f(x) = \log_{10}x$. Describe the transformation that must be applied to $f(x)$ to graph $g(x)$.

a) $g(x) = 3\log_{10}x$

b) $g(x) = \log_{10}(2x)$

c) $g(x) = \log_{10}x - 5$

d) $g(x) = \log_{10}(x + 4)$

2. **a)** State the coordinates of the images of the points $\left(\frac{1}{10}, -1\right)$, $(1, 0)$, and $(10, 1)$ for each of the functions in question 1.

b) State the domain and range of each transformed function, $g(x)$, in question 1.

3. Given the parent function $f(x) = \log_{10}x$, state the equation of the function that results from each of the following pairs of transformations:

a) vertical stretch by a factor of 5, vertical translation 3 units up

b) reflection in the x-axis, horizontal compression by a factor of $\frac{1}{3}$

c) horizontal translation 4 units left, vertical translation 3 units down

d) reflection in the x-axis, horizontal translation 4 units right

PRACTISING

4. Let $f(x) = \log_{10}x$. For each function $g(x)$

K **a)** state the transformations that must be applied to f to produce the graph of g.

b) State the coordinates of the points on g that are images of the points (1, 0) and (10, 1) on the graph of f.

c) State the equation of the asymptote.

d) State the domain and range.

i) $g(x) = -4\log_{10}x + 5$

ii) $g(x) = \frac{1}{2}\log_{10}(x - 6) + 3$

iii) $g(x) = \log_{10}(3x) - 4$

iv) $g(x) = 2\log_{10}[-2(x + 2)]$

v) $g(x) = \log_{10}(2x + 4)$

vi) $g(x) = \log_{10}(-x - 2)$

5. Sketch the graph of each function using transformations. State the domain and range.

a) $f(x) = 3\log_{10}x + 3$

b) $g(x) = -\log_{10}(x - 6)$

c) $h(x) = \log_{10}2x$

d) $j(x) = \log_{10}0.5x - 1$

e) $k(x) = 4\log_{10}\left(\frac{1}{6}x\right) - 2$

f) $r(x) = \log_{10}(-2x - 4)$

6. Compare the functions $f(x) = 10^{\left(\frac{x}{3}\right)} + 1$ and $g(x) = 3\log_{10}(x - 1)$.

7. a) Describe how the graphs of $f(x) = \log_3x$, $g(x) = \log_3(x + 4)$, and $h(x) = \log_3x + 4$ are similar yet different, without drawing the graphs.

b) Describe how the graphs of $f(x) = \log_3x$, $m(x) = 4\log_3x$, and $n(x) = \log_34x$ are similar yet different, without drawing the graphs.

8. The function $f(x) = \log_{10}x$ has the point (10, 1) on its graph.
A If $f(x)$ is vertically stretched by a factor of 3, reflected in the x-axis, horizontally stretched by a factor of 2, horizontally translated 5 units to the right, and vertically translated 2 units up, determine

a) the equation of the transformed function

b) the coordinates of the image point transformed from (10, 1)

c) the domain and range of the transformed function

9. State the transformations that are needed to turn $y = 4\log_{10}(x - 4)$
T into $y = -2\log_{10}(x + 1)$.

10. Describe three characteristics of the function $y = \log_{10}x$ that remain
C unchanged under the following transformations: a vertical stretch by a factor of 4 and a horizontal compression by a factor of 2.

Extending

11. Sketch the graph of $f(x) = \frac{-2}{\log_2(x + 2)}$.