8.3 Evaluating Logarithms

YOU WILL NEED

- graphing calculator

GOAL

Evaluate logarithmic expressions, and approximate the logarithm of a number to any base.

LEARN ABOUT the Math

Jackson knows that a rumour spreads very quickly. He tells three people a rumour. By the end of the next hour, each of these people has told three more people. Each person who hears the rumour tells three more people in the next hour. Jackson has written an algebraic model, $N(t) = 3^{t+1}$, to represent the number of people who hear the rumour within a particular hour, where $N(t)$ is the number of people told during hour t and $t = 1$ corresponds to the hour during which the first three people heard the rumour and started telling others.

? In which hour will an additional 2187 people hear the rumour?

EXAMPLE 1 Selecting a strategy to solve a problem

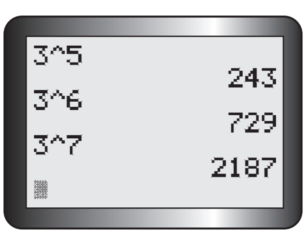
Determine the hour in which an additional 2187 people will hear the rumour.

Solution A: Using a guess-and-check strategy to solve an exponential equation

$N(t) = 2187$

$2187 = 3^{t+1}$

Substitute 2187 for $N(t)$ in the equation.
It is easier to solve the equation if both sides are written as powers with the same base. Using guess and check, write 2187 as a power of 3.



$3^7 = 3^{t+1}$

$7 = t + 1$

$6 = t$

Both sides will be equal when both powers of 3 have the same exponent. Equate the exponents, and solve for t.

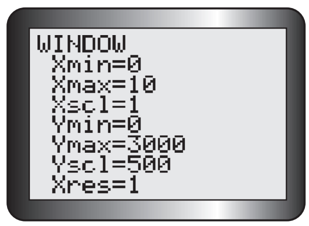
Another 2187 students will hear the rumour during the 6th hour.

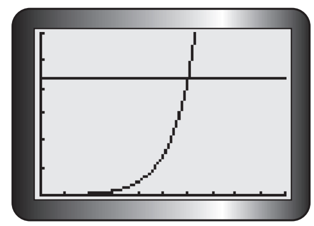
Solution B: Using a graphing calculator to solve an exponential equation

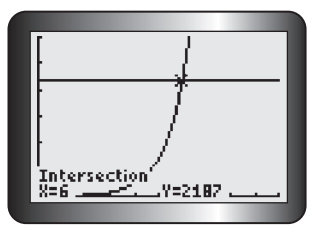
$N(t) = 2187$

$2187 = 3^{t+1}$

A graph can be used to solve the equation.
Enter $y = 3^{x+1}$ in Y1 of the equation editor and $y = 2817$ in Y2. Graph using a window that corresponds to the domain and range in this situation.







The point of intersection for the two functions is the solution to the equation. Use the intersect operation to determine this point.

Another 2187 students will hear the rumour during the 6th hour.

Solution C: Rewriting an exponential equation in logarithmic form

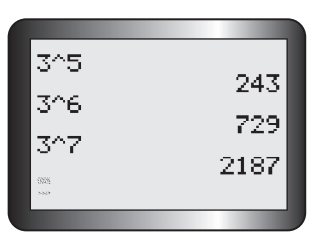
$N(t) = 2187$

$2187 = 3^{t+1}$

Determine the value of the exponent t, when $N(t) = 2187$. To solve for t, rewrite the equation in logarithmic form.

$t + 1 = \log_3 2187$

Since a logarithm is an exponent, evaluate $\log_3 2187$ by determining the exponent to which the base 3 must be raised to get 2187. Use guess and check.



$t + 1 = 7$

$t = 6$

Another 2187 students will hear the rumour during the 6th hour.

Reflecting

A. Solutions A and B used the exponential form of the model, but different strategies. Which one of these strategies will only work for some equations? Explain why.

B. Solution C used the logarithmic form of the model. Is there any advantage of rewriting the model in this form? Explain.

C. If you had to solve the equation $3^{t+1} = 1000$, which strategy would you use? Explain your reasons.

APPLY the Math

EXAMPLE 2 Using reasoning to evaluate logarithmic expressions

Use the definition of a logarithm to determine the value of each expression.

a) $\log_4 64$

b) $\log_3\left(\frac{1}{27}\right)$

c) $\log_2(-4)$

d) $\log_5 \sqrt[3]{25}$

Solution

a) $\log_4 64 = x$ — Determine the exponent to which 4 must be raised to get 64.

$4^x = 64$ — Rewrite the equation in exponential form.

$4^x = 4^3$ — Rewrite 64 as a power of 4.

$x = 3$ — The exponent is 3.

b) $\log_3\left(\frac{1}{27}\right) = x$ — Determine the exponent to which 3 must be raised to get $\frac{1}{27}$.

$3^x = \frac{1}{27}$ — Rewrite the equation in exponential form.

$3^x = 3^{-3}$ — Since $\frac{1}{27} = \frac{1}{3^3}$, $\frac{1}{27}$ can be replaced with 3^{-3}.

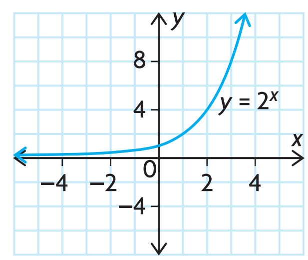
$x = -3$ — The exponent is -3.

c) $\log_2(-4) = x$ — Determine the exponent to which 2 must be raised to get -4.

$2^x = -4$ — Rewrite the equation in exponential form.

There is no solution. — Since 2 is a positive number, there will never be a negative result when 2 is raised to an exponent. The domain of any logarithmic function is $x > 0$.

Recall that the range of $y = 2^x$ is $\{y \in \mathbf{R} \mid y > 0\}$.



d) $\log_5\sqrt[3]{25} = x$ — Determine the exponent to which 5 must be raised to get $\sqrt[3]{25}$.

$5^x = \sqrt[3]{25}$ — Rewrite the radical using the equivalent fractional exponent.

$5^x = \sqrt[3]{5^2}$

$5^x = 5^{\frac{2}{3}}$

$x = \dfrac{2}{3}$ — The exponent is $\frac{2}{3}$.

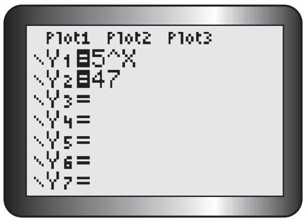
EXAMPLE 3 Selecting a strategy to estimate the logarithm of a number

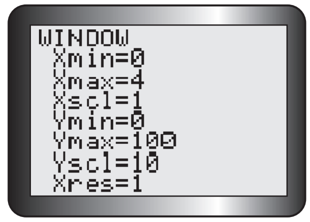
Determine the approximate value of $\log_5 47$.

Solution A: Using graphing technology

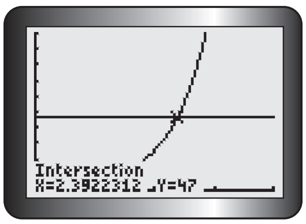
$\log_5 47 = x$ — Determine the exponent to which 5 must be raised to get 47.

$5^x = 47$ — Rewrite the equation in exponential form.





Graph the functions $y = 5^x$ and $y = 47$ using a suitable window.



Determine the point of intersection to estimate the value of x.

$x \doteq 2.39$

Solution B: Using guess and check

$\log_5 47 = x$

$5^x = 47$ — Rewrite the equation in exponential form.

$5^2 = 25$ and $5^3 = 125$ — The exponent must be between 2 and 3.

$5^{2.5} \doteq 55.9$ — Try 2.5. The result is too high.

$5^{2.25} \doteq 37.38$ — Try halfway between 2 and 2.5. The result is too low.

$5^{2.375} \doteq 45.71$ — Try halfway between 2.25 and 2.5. The result is getting close.

$5^{2.4} \doteq 47.59$ — Next try 2.4. The result is a little bit too high.

$5^{2.3875} \doteq 46.64$ — Average 2.4 and 2.375. The result is very close.

$5^{2.39375} \doteq 47.12$ — Refine the guess by averaging 2.4 and 2.3875.

The value is approximately 2.39.

Tech *Support*

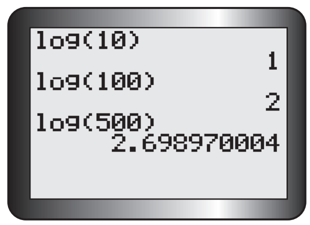
For help using the graphing calculator to find points of intersection, see Technical Appendix, T-12.

EXAMPLE 4 Selecting a strategy to evaluate common logarithms

Use the log key on a calculator to evaluate the following logarithms. Explain how the calculator determined the values.

a) log 10 **b)** log 100 **c)** log 500

Solution



Notice that no base is given with the logarithms. Recall that $\log x = \log_{10} x$.

a) $\log_{10} 10 = x$

$10^x = 10$, so $x = 1$

b) $\log_{10} 100 = x$

$10^x = 100$, so $x = 2$

c) $\log_{10} 500 = x$

$10^x = 500$, so $x \doteq 2.7$

Let x represent the value of each expression. Rewrite each equation in exponential form.

The calculator determined the exponents that must be applied to base 10 to get 10, 100, and 500.

EXAMPLE 5 Examining some general properties of logarithms

Evaluate each of the following logarithms.

a) $\log_6 1$ **b)** $\log_5 5^x$ **c)** $6^{\log_6 x}$

Solution

a) $\log_6 1 = 0$

The value of the expression is the exponent to which 6 must be raised to get 1. A power equals 1 only when its exponent is 0.

$\log_6 1 = x$

$6^x = 1$

$6^x = 6^0$

$x = 0$

To verify, let the expression equal x and rewrite the expression in exponential form.

b) $\log_5 5^x = x$

The value of the expression is the exponent to which 5 must be raised to get 5^x. The exponent must be x.

$$\log_5 5^x = y$$
$$5^y = 5^x$$
$$y = x$$
$$\log_5 5^x = x$$

To verify, let the expression equal y and rewrite the expression in exponential form.

c) $6^{\log_6 x}$

This expression is written in exponential form. Let the expression equal y, and rewrite it in logarithmic form.

$$6^{\log_6 x} = y$$
$$\log_6 y = \log_6 x$$
$$y = x$$
$$6^{\log_6 x} = x$$

The left side equals the right side only if x and y are equal.

In Summary

Key Ideas

- Simple exponential equations can be solved using a variety of strategies:
  - expressing both sides as powers with a common base and then equating the exponents
  - graphing both sides of the equation using graphing technology and then determining the point of intersection
  - rewriting the equation in logarithmic form and simplifying
- A logarithm is an exponent. The logarithm of a number to a given base is the exponent to which the base must be raised to get the number.

Need to Know

- Logarithms of negative numbers do not exist, because a negative number cannot be written as a power of a positive base.
- A logarithm written with any base can be estimated with a calculator, using graphing technology, or guess and check.
- The expression $\log x$ is called a common logarithm. It means $\log_{10} x$, and it can be evaluated using the log key on a calculator.
- The following are some properties of logarithms, where $a > 0$ and $a \neq 1$:
  - $\log_a 1 = 0$
  - $\log_a a^x = x$
  - $a^{\log_a x} = x$

CHECK *Your Understanding*

1. Express in logarithmic form.

a) $4^2 = 16$

b) $3^4 = 81$

c) $8^0 = 1$

d) $6^{-2} = \frac{1}{36}$

e) $\left(\frac{1}{3}\right)^3 = \frac{1}{27}$

f) $8^{\frac{1}{3}} = 2$

2. Express in exponential form.

a) $\log_2 8 = 3$

b) $\log_5 \frac{1}{25} = -2$

c) $\log_3 81 = 4$

d) $\log_{\frac{1}{6}} 216 = -3$

e) $\log_6 \sqrt{6} = \frac{1}{2}$

f) $\log_{10} 1 = 0$

3. Evaluate.

a) $\log_5 5$

b) $\log_7 1$

c) $\log_2\left(\frac{1}{4}\right)$

d) $\log_7 \sqrt{7}$

e) $\log_{\frac{2}{3}}\left(\frac{8}{27}\right)$

f) $\log_2 \sqrt[3]{2}$

PRACTISING

4. Solve for x. Round your answers to two decimal places, if necessary.

a) $\log\left(\frac{1}{10}\right) = x$

b) $\log 1 = x$

c) $\log(1\,000\,000) = x$

d) $\log 25 = x$

e) $\log x = 0.25$

f) $\log x = -2$

5. Evaluate.

a) $\log_6 \sqrt{6}$

b) $\log_5 125 - \log_5 25$

c) $\log_3 81 + \log_4 64$

d) $\log_2 \frac{1}{4} - \log_3 1$

e) $\log_5 \sqrt[3]{5}$

f) $\log_3 \sqrt{27}$

6. **K** Use your knowledge of logarithms to solve each of the following equations for x.

a) $\log_5 x = 3$

b) $\log_x 27 = 3$

c) $\log_4 \frac{1}{64} = x$

d) $\log_{\frac{1}{4}} x = -2$

e) $\log_5 x = \frac{1}{2}$

f) $\log_4 x = 1.5$

7. Graph $f(x) = 3^x$. Use your graph to estimate each of the following logarithms.

a) $\log_3 17$ **b)** $\log_3 36$ **c)** $\log_3 112$ **d)** $\log_3 143$

8. Estimate the value of each of the following logarithms to two decimal places.

a) $\log_4 32$ **b)** $\log_6 115$ **c)** $\log_3 212$ **d)** $\log_{11} 896$

9. Evaluate.

a) $\log_3 3^5$

b) $5^{\log_5 25}$

c) $4^{\log_4 \frac{1}{16}}$

d) $\log_m m^n$

e) $a^{\log_a b}$

f) $\log_{\frac{1}{10}} 1$

10. Evaluate $\log_2 16^{\frac{1}{3}}$.

11. **A** The number of mold spores in a petri dish increases by a factor of 10 every week. If there are initially 40 spores in the dish, how long will it take for there to be 2000 spores?

12. **Half-life** is the time it takes for half of a sample of a radioactive element to decay. The function $M(t) = P\left(\frac{1}{2}\right)^{\frac{t}{h}}$ can be used to calculate the mass remaining if the half-life is h and the initial mass is P. The half-life of radium is 1620 years.

a) If a laboratory has 5 g of radium, how much will there be in 150 years?

b) How many years will it take until the laboratory has only 4 g of radium?

13. The function $s(d) = 0.159 + 0.118 \log d$ relates the slope, s, of a beach to the average diameter, d, in millimetres, of the sand particles on the beach. Which beach has a steeper slope: beach A, which has very fine sand with $d = 0.0625$, or beach B, which has very coarse sand with $d = 1$? Justify your decision.

14. The function $S(d) = 93 \log d + 65$ relates the speed of the wind, S, in miles per hour, near the centre of a tornado to the distance that the tornado travels, d, in miles.

a) If a tornado travels a distance of about 50 miles, estimate its wind speed near its centre.

b) If a tornado has sustained winds of approximately 250 mph, estimate the distance it can travel.

15. The astronomer Johannes Kepler (1571–1630) determined that the time, D, in days, for a planet to revolve around the Sun is related to the planet's average distance from the Sun, k, in millions of kilometres. This relation is defined by the equation $\log D = \frac{3}{2} \log k - 0.7$. Verify that Kepler's equation gives a good approximation of the time it takes for Earth to revolve around the Sun, if Earth is about 150 000 000 km from the Sun.

16. Use Kepler's equation from question 15 to estimate the period of revolution of each of the following planets about the Sun, given its distance from the Sun.

a) Uranus, 2854 million kilometres

b) Neptune, 4473 million kilometres

17. The doubling function $y = y_0 2^{\frac{t}{D}}$ can be used to model exponential growth when the doubling time is D. The bacterium *Escherichia coli* has a doubling period of 0.32 h. A culture of *E. coli* starts with 100 bacteria.
T

a) Determine the equation for the number of bacteria, y, in x hours.
b) Graph your equation.
c) Graph the inverse.
d) Determine the equation of the inverse. What does this equation represent?
e) How many hours will it take for there to be 450 bacteria in the culture? Explain your strategy.

18. To evaluate a logarithm whose base is not 10 you can use the following relationship (which will be developed in section 8.5):

$$\log_a b = \frac{\log b}{\log a}$$

Use this to evaluate each of the following to four decimal places.

a) $\log_5 5$
b) $\log_2 10$
c) $\log_5 45$
d) $\log_8 92$
e) $\log_4 0.5$
f) $\log_7 325$

19. Consider the expression $\log_5 a$.
C

a) For what values of a will this expression yield positive numbers?
b) For what values of a will this expression yield negative numbers?
c) For what values of a will this expression be undefined?

Extending

20. Simplify.

a) $3^{\log_3 27} + 10^{\log_{10} 1000}$
b) $5^{\log_5 8} - 3^{\log_3 5 + \log_3 7}$

21. Determine the inverse of each relation.

a) $y = \sqrt[3]{x}$
b) $y = 3(2)^x$
c) $y = (0.5)^{x+2}$
d) $y = 3\log_2(x - 3) + 2$

22. Graph each function and its inverse. State the domain, range, and asymptote of each. Determine the equation of the inverse.

a) $y = 3\log(x + 6)$
b) $y = -2\log_5 3x$
c) $y = 2 + 3\log x$
d) $y = 20(8)^x$
e) $y = 2(3)^{x+2}$
f) $y = -5^x - 3$

23. For the function $y = \log_{10} x$, where $0 < x < 1000$, how many integer values of y are possible if $y > -20$?