8.4 Laws of Logarithms

GOAL

Recognize the connection between the laws of exponents and the laws of logarithms, and use the laws of logarithms to simplify expressions.

LEARN ABOUT the Math

Since the logarithm function with base a is the inverse of the exponential function with base a, it makes sense that each exponent law should have a corresponding logarithmic law. You have seen that the exponential property $a^0 = 1$ has the corresponding logarithmic property $\log_a 1 = 0$.

Recall the following exponent laws:

- product law: $a^x \times a^y = a^{x+y}$
- quotient law: $a^x \div a^y = a^{x-y}$
- power law: $(a^x)^y = a^{xy}$

? What are the corresponding laws of logarithms for these exponent laws?

EXAMPLE 1 Connecting the product laws

Determine an equivalent expression for $\log_a(mn)$, where a, m, and n are positive numbers and $a \neq 1$.

Solution

Let $m = a^x$ and $n = a^y$. ← Since a, m, and n are all positive, m and n can be expressed as powers of a.

$mn = (a^x)(a^y) = a^{x+y}$ ← Substitute the expressions for m and n into the product mn. Simplify using the product law for exponents.

$\log_a(mn) = \log_a(a^{x+y})$ ← These expressions must be equal since $mn = a^{x+y}$, as shown above. On the right side of this equation, the exponent that must be applied to a to get a^{x+y} is $x + y$.

$\log_a(mn) = x + y$

$m = a^x$ so $\log_a m = x$

$n = a^y$ so $\log_a n = y$

Write the powers involving m and n in logarithmic form. Substitute the logarithmic expressions into the equation $\log_a(mn) = x + y$.

$\log_a(mn) = \log_a m + \log_a n$

The logarithm of a product is equal to the sum of the logarithms of the factors.

EXAMPLE 2 Connecting the quotient laws

Determine an equivalent expression for $\log_a\left(\frac{m}{n}\right)$, where a, m, and n are positive numbers and $a \neq 1$.

Solution

Let $m = a^x$ and $n = a^y$.

Since a, m, and n are all positive, m and n can be expressed as powers of a.

$\frac{m}{n} = \frac{a^x}{a^y} = a^{x-y}$

Substitute the expression for m and n into the quotient $\frac{m}{n}$. Simplify using the quotient law for exponents.

$\log_a\left(\frac{m}{n}\right) = \log_a(a^{x-y})$

$\log_a\left(\frac{m}{n}\right) = x - y$

These expressions must be equal since $\frac{m}{n} = a^{x-y}$, as shown above. On the right side of this equation, the exponent that must be applied to a to get a^{x-y} is $x - y$.

$m = a^x$ so $\log_a m = x$

$n = a^y$ so $\log_a n = y$

Write the powers involving m and n in logarithmic form. Substitute the logarithmic expressions into the equation $\log_a\left(\frac{m}{n}\right) = x - y$.

$\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$

The logarithm of a quotient is equal to the logarithm of the dividend minus the logarithm of the divisor.

EXAMPLE 3 Connecting the power laws

Determine an equivalent expression for $\log_a(m^n)$, where a, m, and n are positive numbers and $a \neq 1$.

Solution

Let $m = a^x$. ← Since a and m are positive, m can be expressed as a power of a.

$m^n = (a^x)^n = a^{nx}$ ← Substitute the expression for m into the power m^n. Simplify using the power law for exponents.

$\log_a(m^n) = \log_a(a^{nx})$
$\log_a(m^n) = nx$ ← These expressions must be equal since $m^n = a^{nx}$, as shown above. On the right side of this equation, the exponent that must be applied to a to get a^{nx} is nx.

$m = a^x$, so $\log_a m = x$ ← Write the power involving m in logarithmic form. Substitute the logarithmic expressions into the equation $\log_a(m^n) = nx$.

$\log_a(m^n) = n\log_a m$ ← The logarithm of a power of a number is equal to the exponent multiplied by the logarithm of the number.

Reflecting

A. Which exponent law is related to each logarithm law? How can this be seen in the operations used in each pair of related laws?

B. Why does it make sense that each exponent law has a related logarithm law?

C. Can $\log_2 5 + \log_3 7$ be expressed as a single logarithm using any of the logarithm laws? Explain.

D. Can $\log_6 12 - \log_4 8$ be expressed as a single logarithm using any of the logarithm laws? Explain.

APPLY the Math

EXAMPLE 4 Selecting strategies to simplify logarithmic expressions

Simplify each logarithmic expression.

a) $\log_3 6 + \log_3 4.5$ **b)** $\log_2 48 - \log_2 3$ **c)** $\log_5 \sqrt[3]{25}$

Solution

Communication *Tip*

The laws of logarithms are generalizations that simplify the calculation of logarithms with the same base, much like the laws of exponents simplify the calculation of powers with the same base. The laws of logarithms and the laws of exponents can be used both forward and backward to simplify and evaluate expressions.

a) $\log_3 6 + \log_3 4.5$ — Since the logarithms have the same base, the sum can be simplified.

$= \log_3(6 \times 4.5)$ — The sum of the logarithms of two numbers is the logarithm of their product.

$= \log_3 27$ — The exponent that must be applied to 3 to get 27 is 3.

$= 3$

b) $\log_2 48 - \log_2 3$ — These logarithms have the same base, so the difference of the logarithms of the two numbers can be written as the logarithm of their quotient.

$= \log_2\left(\frac{48}{3}\right)$

$= \log_2 16$ — The exponent that must be applied to 2 to get 16 is 4.

$= 4$

c) $\log_5 \sqrt[3]{25}$

$= \log_5 25^{\frac{1}{3}}$ — Change the cube root into a rational exponent.

$= \frac{1}{3}\log_5 25$ — The logarithm of a power is the same as the exponent multiplied by the logarithm of the base of the power.

$= \frac{1}{3} \times 2$ — Evaluate $\log_5 25$, and then multiply the result by the fraction.

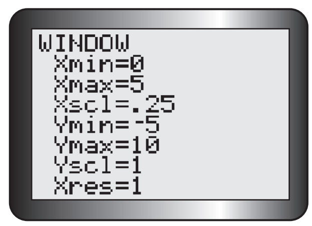
$= \frac{2}{3}$

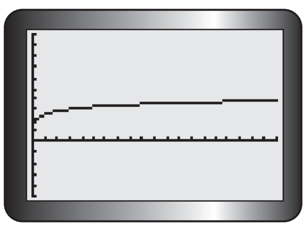
EXAMPLE 5 Connecting laws of logarithms to graphs of logarithmic functions

Graph the functions $f(x) = \log(1000x)$ and $g(x) = 3 + \log x$. How do the graphs compare? Explain your findings algebraically.

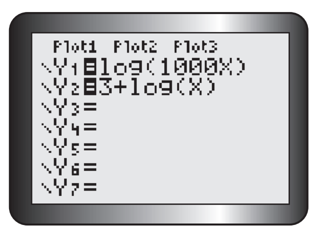
Solution

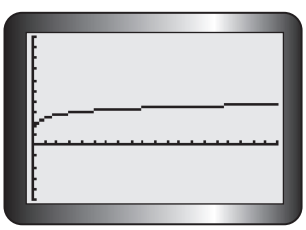
Graph the function $f(x)$ in Y1 with a graphing calculator, using the following window settings.





Add the function $g(x)$ in Y2 using the same window. The two graphs are identical on the screen.





The graphs are equivalent.

$f(x) = \log(1000x)$ — Notice that $\log(1000x)$ is the logarithm of a product.

$= \log 1000 + \log x$ — Rewrite the logarithm of the product as the sum of the logarithms of the factors.

$= 3 + \log x$ — Evaluate log 1000.

$= g(x)$ — The result is equivalent to the function $g(x)$.

EXAMPLE 6 Selecting strategies to simplify logarithmic expressions

Use the properties of logarithms to express $\log_a\sqrt{\frac{x^3y^2}{w}}$ in terms of $\log_a x$, $\log_a y$, and $\log_a w$.

Solution

$$\log_a\sqrt{\frac{x^3y^2}{w}} = \log_a\left(\frac{x^3y^2}{w}\right)^{\frac{1}{2}}$$

Express the square root using the rational exponent of $\frac{1}{2}$.

$$= \frac{1}{2}\log_a\left(\frac{x^3y^2}{w}\right)$$

Use the power law of logarithms to write an equivalent expression.

$$= \frac{1}{2}(\log_a x^3y^2 - \log_a w)$$

Express the logarithm of the quotient of x^3y^2 and w as a difference.

$$= \frac{1}{2}(\log_a x^3 + \log_a y^2 - \log_a w)$$

Express the logarithm of the product of x^3y^2 as a sum.

$$= \frac{1}{2}\log_a x^3 + \frac{1}{2}\log_a y^2 - \frac{1}{2}\log_a w$$

Expand using the distributive property.

$$= \frac{1}{2} \times 3\log_a x + \frac{1}{2} \times 2\log_a y - \frac{1}{2}\log_a w$$

Use the power law of logarithms again to write an equivalent expression where appropriate.

$$= \frac{3}{2}\log_a x + \log_a y - \frac{1}{2}\log_a w$$

Simplify.

In Summary

Key Ideas

- The laws of logarithms are directly related to the laws of exponents, since logarithms are exponents.
- The laws of logarithms can be used to simplify logarithmic expressions if all the logarithms have the same base.

Need to Know

- The laws of logarithms are as follows, where $a > 0$, $x > 0$, $y > 0$, and $a \neq 1$:
  - **product law of logarithms:** $\log_a xy = \log_a x + \log_a y$
  - **quotient law of logarithms:** $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$
  - **power law of logarithms:** $\log_a x^r = r\log_a x$

CHECK *Your Understanding*

1. Write each expression as a sum or difference of logarithms.

a) $\log(45 \times 68)$

b) $\log_m pq$

c) $\log\left(\frac{123}{31}\right)$

d) $\log_m\left(\frac{p}{q}\right)$

e) $\log_2(14 \times 9)$

f) $\log_4\left(\frac{81}{30}\right)$

2. Express each of the following as a logarithm of a product or quotient.

a) $\log 5 + \log 7$

b) $\log_3 4 - \log_3 2$

c) $\log_m a + \log_m b$

d) $\log x - \log y$

e) $\log_6 7 + \log_6 8 + \log_6 9$

f) $\log_4 10 + \log_4 12 - \log_4 20$

3. Express each of the following in the form $r \log_a x$.

a) $\log 5^2$

b) $\log_3 7^{-1}$

c) $\log_m p^q$

d) $\log \sqrt[3]{45}$

e) $\log_7(36)^{0.5}$

f) $\log_5 \sqrt[5]{125}$

PRACTISING

4. **K** Use the laws of logarithms to simplify and then evaluate each expression.

a) $\log_3 135 - \log_3 5$

b) $\log_5 10 + \log_5 2.5$

c) $\log 50 + \log 2$

d) $\log_4 4^7$

e) $\log_2 224 - \log_2 7$

f) $\log \sqrt{10}$

5. Describe how the graphs of $y = \log_2(4x)$, $y = \log_2(8x)$, and $y = \log_2\left(\frac{x}{2}\right)$ are related to the graph of $y = \log_2 x$.

6. Evaluate the following logarithms.

a) $\log_{25} 5^3$

b) $\log_6 54 + \log_6 2 - \log_6 3$

c) $\log_6 6\sqrt{6}$

d) $\log_2 \sqrt{36} - \log_2 \sqrt{72}$

e) $\log_3 54 + \log_3\left(\frac{3}{2}\right)$

f) $\log_8 2 + 3\log_8 2 + \frac{1}{2}\log_8 16$

7. Use the laws of logarithms to express each of the following in terms of $\log_b x$, $\log_b y$, and $\log_b z$.

a) $\log_b xyz$

b) $\log_b\left(\frac{z}{xy}\right)$

c) $\log_b x^2y^3$

d) $\log_b \sqrt{x^5yz^3}$

8. Explain why $\log_5 3 + \log_5 \frac{1}{3} = 0$.

9. Write each expression as a single logarithm.

a) $3 \log_5 2 + \log_5 7$

b) $2 \log_3 8 - 5 \log_3 2$

c) $2 \log_2 3 + \log_2 5$

d) $\log_3 12 + \log_3 2 - \log_3 6$

e) $\log_4 3 + \frac{1}{2} \log_4 8 - \log_4 2$

f) $2 \log 8 + \log 9 - \log 36$

10. **A** Use the laws of logarithms to express each side of the equation as a single logarithm. Then compare both sides of the equation to solve.

a) $\log_2 x = 2 \log_2 7 + \log_2 5$

b) $\log x = 2 \log 4 + 3 \log 3$

c) $\log_4 x + \log_4 12 = \log_4 48$

d) $\log_7 x = 2 \log_7 25 - 3 \log_7 5$

e) $\log_3 x = 2 \log_3 10 - \log_3 25$

f) $\log_5 x - \log_5 8 = \log_5 6 + 3 \log_5 2$

11. Write each expression as a single logarithm. Assume that all the variables represent positive numbers.

a) $\log_2 x + \log_2 y + \log_2 z$

b) $\log_5 u - \log_5 v + \log_5 w$

c) $\log_6 a - (\log_6 b + \log_6 c)$

d) $\log_2 x^2 - \log_2 xy + \log_2 y^2$

e) $1 + \log_3 x^2$

f) $3 \log_4 x + 2 \log_4 x - \log_4 y$

12. Write $\frac{1}{2} \log_a x + \frac{1}{2} \log_a y - \frac{3}{4} \log_a z$ as a single logarithm. Assume that all the variables represent positive numbers.

13. Describe the transformations that take the graph of $f(x) = \log_2 x$ to the graph of $g(x) = \log_2(8x^3)$.

14. **T** Use different expressions to create two logarithmic functions that have the same graph. Demonstrate algebraically why these functions have the same graph.

15. **C** Explain how the laws of logarithms can help you evaluate $\log_3\left(\frac{\sqrt[5]{27}}{2187}\right)$.

Extending

16. Explain why $\log_x x^{m-1} + 1 = m$.

17. If $\log_b x = 0.3$, find the value of $\log_b x\sqrt{x}$.

18. Use graphing technology to draw the graphs of $y = \log x + \log 2x$ and $y = \log 2x^2$. Although the graphs are different, simplifying the first expression using the laws of logarithms produces the second expression. Explain why the graphs are different.

19. Create a pair of equivalent expressions that demonstrate each of the laws of logarithms. Prove that these expressions are equivalent.