8.5 Solving Exponential Equations

GOAL

Solve exponential equations in one variable using a variety of strategies.

LEARN ABOUT the Math

All radioactive substances decrease in mass over time.

Jamie works in a laboratory that uses radioactive substances. The laboratory received a shipment of 200 g of radioactive radon, and 16 days later, 12.5 g of the radon remained.

? What is the half-life of radon?

EXAMPLE 1 Selecting a strategy to solve an exponential equation

Calculate the half-life of radon.

Solution A: Solving the equation algebraically by writing both sides with the same base

$$M(t) = P\left(\frac{1}{2}\right)^{\frac{t}{h}}$$

Write the formula for half-life, where h is the half-life period.

$$12.5 = 200\left(\frac{1}{2}\right)^{\frac{16}{h}}$$

Substitute the known values $M(t)$ (for the remaining mass), P (the original mass), and t (the time in days).

$$\frac{12.5}{200} = \left(\frac{1}{2}\right)^{\frac{16}{h}}$$

Isolate h by dividing both sides of the equation by 200.

$$\frac{1}{16} = \left(\frac{1}{2}\right)^{\frac{16}{h}}$$

Express the left side of the equation as a fraction.

$$\left(\frac{1}{2}\right)^{4} = \left(\frac{1}{2}\right)^{\frac{16}{h}}$$

Both sides can be written with the same base.

$$4 = \frac{16}{h}$$

The powers are equal when the exponents are equal. Solve the resulting equation.

$$4h = 16$$
$$h = 4$$

Multiply both sides by h, and then divide by 4 to solve for h.

The half-life of radon is 4 days.

Solution B: Solving the equation algebraically by taking the logarithm of both sides

$$M(t) = P\left(\frac{1}{2}\right)^{\frac{t}{h}}$$

Write the formula for half-life.

$$12.5 = 200\left(\frac{1}{2}\right)^{\frac{16}{h}}$$

Substitute the given values.

$$\frac{12.5}{200} = \left(\frac{1}{2}\right)^{\frac{16}{h}}$$

Divide both sides of the equation by 200.

$$0.0625 = (0.5)^{\frac{16}{h}}$$

Express the fractions as decimals.

$$\log(0.0625) = \log(0.5)^{\frac{16}{h}}$$

If two quantities are equal, then the logs of the quantities will also be equal.

$$\log(0.0625) = \frac{16}{h}\log(0.5)$$

Use the power rule for logarithms to rewrite the right side of the equation without an exponent.

$$h\log(0.0625) = 16\log(0.5)$$

Multiply both sides by h.

$$h = \frac{16\log(0.5)}{\log(0.0625)}$$
$$h = 4$$

Divide both sides of the equation by $\log(0.0625)$ and evaluate the result with a calculator.

The half-life of radon is 4 days.

Solution C: Solving the equation graphically using graphing technology

$$M(t) = P\left(\frac{1}{2}\right)^{\frac{t}{h}}$$

Write the formula for half-life.

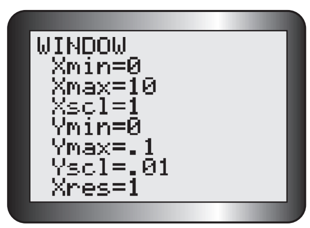
$$12.5 = 200\left(\frac{1}{2}\right)^{\frac{16}{h}}$$

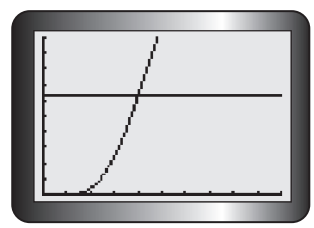
Substitute the given values.

$$\frac{12.5}{200} = \left(\frac{1}{2}\right)^{\frac{16}{h}}$$

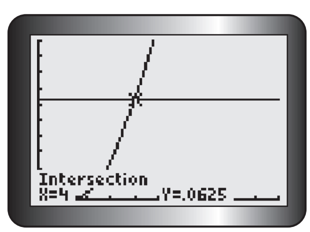
To solve for h, divide both sides of the equation by 200.

$$0.0625 = (0.5)^{\frac{16}{h}}$$





Express the fractions as decimals. Enter the right side of the equation into Y1 of the equation editor. Enter the left side into Y2. Graph using a window that corresponds to the domain and range in this context.



Determine the point of intersection of the graphs using the intersect operation. The x-coordinate of the point of intersection is the solution to the equation.

The half-life of radon is 4 days.

Tech Support

For help using the graphing calculator to determine points of intersection, see Technical Appendix, T-12.

Reflecting

A. Why did the strategy that was used in Solution A result in an exact answer? Will this strategy always result in an exact answer? Explain.

B. Which of the strategies used in the three different solutions will always result in an exact answer? Explain.

C. Which of the three strategies do you prefer? Justify your preference.

APPLY the Math

EXAMPLE 2 Selecting a strategy to solve an exponential equation with more than one power

Solve $2^{x+2} - 2^x = 24$.

Solution

$$2^{x+2} - 2^x = 24$$

The terms on the left side of the equation cannot be combined.

$$2^x(2^2 - 1) = 24$$

Divide out the common factor of 2^x on the left side of the equation.

$$\begin{aligned} 2^x(4 - 1) &= 24 \\ 2^x(3) &= 24 \\ 2^x &= 8 \\ 2^x &= 2^3 \\ x &= 3 \end{aligned}$$

Simplify the expression in brackets.
Divide both sides by 3.
Express the right side of the equation as a power of 2.

EXAMPLE 3 Using logarithms to solve a problem

An investment of \$2500 grows at a rate of 4.8% per year, compounded annually. How long will it take for the investment to be worth \$4000? Recall that the formula for compound interest is $A = P(1 + i)^n$.

Solution

$$\begin{aligned} A &= P(1 + i)^n \\ 4000 &= 2500(1.048)^n \end{aligned}$$

Substitute the known values ($P = 2500$, $i = 0.048$, and $A = 4000$) into the formula. The variable n represents the number of years.

$$\frac{4000}{2500} = (1.048)^n$$

Divide both sides of the equation by 2500.

$$1.6 = (1.048)^n$$

Express the result as a decimal.

$$\log(1.6) = \log(1.048)^n$$

Take the log of both sides to solve for n.

$$\log(1.6) = n\log(1.048)$$

Use the power rule for logarithms to rewrite the equation.

$$n = \frac{\log 1.6}{\log(1.048)} \doteq 10.025$$

Divide both sides of the equation by $\log(1.048)$ to solve for n.

It will take approximately 10.025 years for the investment to be worth \$4000.

EXAMPLE 4 Selecting a strategy to solve an exponential equation with different bases

Solve $2^{x+1} = 3^{x-1}$ to three decimal places.

Solution

$$2^{x+1} = 3^{x-1}$$

Both sides of the equation cannot be written with the same base.

$$\log(2^{x+1}) = \log(3^{x-1})$$

Take the log of both sides of the equation.

$$(x+1)\log 2 = (x-1)\log 3$$

Use the power rule for logarithms to rewrite both sides of the equation with no exponents.

$$x\log 2 + \log 2 = x\log 3 - \log 3$$

Expand using the distributive property.

$$\log 2 + \log 3 = x\log 3 - x\log 2$$

Collect like terms to solve the equation.

$$\frac{\log 2 + \log 3}{\log 3 - \log 2} = \frac{x(\log 3 - \log 2)}{\log 3 - \log 2}$$

Divide out the common factor of x on the right side. Then divide both sides by $\log 3 - \log 2$.

$$\frac{\log 2 + \log 3}{\log 3 - \log 2} = x$$

Evaluate using a calculator.

$$4.419 \doteq x$$

Round the answer to the required number of decimal places.

In Summary

Key Ideas

- Two exponential expressions with the same base are equal when their exponents are equal. For example, if $a^m = a^n$, then $m = n$, where $a > 0$, $a \neq 1$, and $m, n \in \mathbf{R}$.
- If two expressions are equal, taking the log of both expressions maintains their equality. For example, if $M = N$, then $\log_a M = \log_a N$, where $M, N > 0$, $a > 0$, $a \neq 1$.

Need to Know

- To solve an exponential equation algebraically, take the logarithm of both sides of the equation using a base of 10, and then use the power rule for logarithms to simplify the equation and solve for the unknown.
- Sometimes an exponential equation can be solved algebraically by writing both sides of the equation with the same base (if possible), setting the exponents equal to each other, and solving for the unknown.
- Exponential equations can also be solved with graphing technology, using the same strategies that are used for other kinds of equations.

CHECK *Your Understanding*

1. Solve.

a) $5^x = 625$
b) $4^{2x} = 2^{5-x}$
c) $9^{x+1} = 27^{2x-3}$
d) $8^{x-1} = \sqrt[3]{16}$
e) $2^{3x} = \frac{1}{2}$
f) $4^{2x} = \frac{1}{16}$

2. Solve. Round your answers to three decimal places.

a) $2^x = 17$
b) $6^x = 231$
c) $30(5^x) = 150$
d) $210 = 40(1.5)^x$
e) $5^{1-x} = 10$
f) $6^{\frac{x}{3}} = 30$

3. Solve by rewriting in exponential form.

a) $x = \log_3 243$
b) $x = \log_6 216$
c) $x = \log_5 5\sqrt{5}$
d) $x = \log_2 \sqrt[5]{8}$
e) $x = \log_2\left(\frac{1}{4}\right)$
f) $x = \log_3\left(\frac{1}{\sqrt{3}}\right)$

PRACTISING

4. The formula to calculate the mass, $M(t)$, remaining from an original sample of radioactive material with mass P, is determined using the formula $M(t) = P\left(\frac{1}{2}\right)^{\frac{t}{h}}$, where t is time and h is the half-life of the substance. The half-life of a radioactive substance is 8 h. How long will it take for a 300 g sample to decay to each mass?

a) 200 g
b) 100 g
c) 75 g
d) 20 g

5. Solve.

K **a)** $49^{x-1} = 7\sqrt{7}$
b) $2^{3x-4} = 0.25$
c) $\left(\frac{1}{4}\right)^{x+4} = \sqrt{8}$
d) $36^{2x+4} = \left(\sqrt{1296}\right)^x$
e) $2^{2x+2} + 7 = 71$
f) $9^{2x+1} = 81(27^x)$

6. A **a)** If \$500 is deposited into an account that pays 8%/a compounded annually, how long will it take for the deposit to double?

b) A \$1000 investment is made in a trust fund that pays 12%/a compounded monthly. How long will it take the investment to grow to \$5000?

c) A \$5000 investment is made in a savings account that pays 10%/a compounded quarterly. How long will it take for the investment to grow to \$7500?

d) If you invested \$500 in an account that pays 12%/a compounded weekly, how long would it take for your deposit to triple?

7. A bacteria culture doubles every 15 min. How long will it take for a culture of 20 bacteria to grow to a population of 163 840?

8. Solve for x.

a) $4^{x+1} + 4^x = 160$

b) $2^{x+2} + 2^x = 320$

c) $2^{x+2} - 2^x = 96$

d) $10^{x+1} - 10^x = 9000$

e) $3^{x+2} + 3^x = 30$

f) $4^{x+3} - 4^x = 63$

9. Choose a strategy to solve each equation, and explain your choice. (Do not solve.)

a) $225(1.05)^x = 450$

b) $3^{x+2} + 3^x = 270$

10. Solve. Round your answers to three decimal places.

a) $5^{t-1} = 3.92$

b) $x = \log_3 25$

c) $4^{2x} = 5^{2x-1}$

d) $x = \log_2 53.2$

11. A plastic sun visor allows light to pass through, but reduces the intensity of the light. The intensity is reduced by 5% if the plastic is 1 mm thick. Each additional millimetre of thickness reduces the intensity by another 5%.

a) Use an equation to model the relation between the thickness of the plastic and the intensity of the light.

b) How thick is a piece of plastic that reduces the intensity of the light to 60%?

12. T Solve $3^{2x} - 5(3^x) = -6$.

13. C If $\log_a x = y$, show that $y = \frac{\log x}{\log a}$. Explain how this relationship could be used to graph $y = \log_5 x$ on a graphing calculator.

Extending

14. Solve for x.

a) $2^{x^2} = 32(2^{4x})$

b) $3^{x^2+20} = \left(\frac{1}{27}\right)^{3x}$

c) $2 \times 3^x = 7 \times 5^x$.

15. If $\log_a 2 = \log_b 8$, show that $a^3 = b$.

16. Determine the point of intersection for the graphs of $y = 3(5^{2x})$ and $y = 6(4^{3x})$. Round your answer to three decimal places.

17. Solve for x, to two decimal places.

a) $6^{3x} = 4^{2x-3}$

b) $(1.2)^x = (2.8)^{x+4}$

c) $3(2)^x = 4^{x+1}$

18. Solve for x, to two decimal places.

$(2^x)^x = 10$