8.6 Solving Logarithmic Equations

GOAL

Solve logarithmic equations with one variable algebraically.

LEARN ABOUT the Math

The Richter scale is used to compare the intensities of earthquakes. The Richter scale magnitude, R, of an earthquake is determined using $R = \log\left(\frac{a}{T}\right) + B$, where a is the amplitude of the vertical ground motion in microns (μ), T is the period of the seismic wave in seconds, and B is a factor that accounts for the weakening of the seismic waves. (1 μ is equivalent to 10^{-6} m.)

? An earthquake measured 5.5 on the Richter scale, and the period of the seismic wave was 1.8 s. If B equals 3.2, what was the amplitude, a, of the vertical ground motion?

EXAMPLE 1 Selecting an algebraic strategy to solve a logarithmic equation

Determine the amplitude, a, of the vertical ground motion.

Solution

$$R = \log\left(\frac{a}{T}\right) + B$$

$$5.5 = \log\left(\frac{a}{1.8}\right) + 3.2$$

Substitute the given values into the equation.

$$2.3 = \log\left(\frac{a}{1.8}\right)$$

Isolate the term with the unknown, a, by subtracting 3.2 from both sides.

$$10^{2.3} = \frac{a}{1.8}$$

Rewrite the equation in exponential form.

$$10^{2.3} \times 1.8 = a$$

Multiply both sides by 1.8 to solve for a.

$$359.1\ \mu = a$$

The amplitude of the vertical ground motion was about $359.1\ \mu$.

To get a better idea of the size of this number, change microns to metres or centimetres. $359.1\ \mu = 0.000\ 359\ 1$ m or 0.035 91 cm.

Reflecting

A. What strategies for solving a linear equation were used to solve this logarithmic equation?

B. Why was the equation rewritten in exponential form?

C. How would the strategies have changed if the value of a had been given and the value of T had to be determined?

APPLY the Math

EXAMPLE 2 Selecting an algebraic strategy to solve a logarithmic equation

Solve.

a) $\log_x 0.04 = -2$ **b)** $\log_7(3x - 5) = \log_7 16$

Solution

a) $\log_x 0.04 = -2$

$x^{-2} = 0.04$

Express the equation in exponential form.

$x^{-2} = \frac{1}{25}$

Rewrite the decimal as a fraction.
$0.04 = \frac{4}{100} = \frac{1}{25}$

$x^{-2} = 5^{-2}$

$x = 5$

Express $\frac{1}{25}$ as a power with exponent -2. Since the exponents are equal, the bases must be equal.

b) $\log_7(3x - 5) = \log_7 16$

If $\log_a M = \log_a N$, then $M = N$.

$3x - 5 = 16$

Since 7 is the base of both logs, the two expressions must be equal.

$3x = 21$

Add 5 to both sides of the equation.

$x = 7$

EXAMPLE 3 Representing sums and differences of logs as single logarithms to solve a logarithmic equation

Solve.

a) $\log_2 30x - \log_2 5 = \log_2 12$ **b)** $\log x + \log x^2 = 12$

Solution

a)
$$\log_2 30x - \log_2 5 = \log_2 12$$
$$\log_2\left(\frac{30x}{5}\right) = \log_2 12$$
$$\log_2(6x) = \log_2 12$$

The logs are written with the same base, so the difference of the logs can be written as the log of a quotient. Simplify.

$$6x = 12$$

Since both sides of the equation are written with the same base, the two expressions are equal.

$$x = 2$$

b)
$$\log x + \log x^2 = 12$$
$$\log(x \times x^2) = 12$$

The logs are written with the same base, so the sum of the logs can be written as the log of a product. Simplify.

$$\log(x^3) = 12$$
$$x^3 = 10^{12}$$

Express the equation in exponential form.

$$\sqrt[3]{x^3} = \sqrt[3]{10^{12}}$$

Take the cube root of both sides to solve for x.

$$x = 10^4$$
$$x = 10\ 000$$

EXAMPLE 4 Selecting a strategy to solve a logarithmic equation that involves quadratics

Solve $\log_2(x + 3) + \log_2(x - 3) = 4$.

Solution

$$\log_2(x + 3) + \log_2(x - 3) = 4$$
$$\log_2(x + 3)(x - 3) = 4$$
$$\log_2(x^2 - 3x + 3x - 9) = 4$$
$$\log_2(x^2 - 9) = 4$$

Since both logarithms have base 2, rewrite the left side as a single logarithm using the product law. Multiply the binomials, and simplify.

$$x^2 - 9 = 2^4$$
$$x^2 - 9 = 16$$
$$x^2 = 25$$

Rewrite the equation in exponential form and solve for x^2.

$$x = \pm\sqrt{25}$$

Take the square root of both sides. There are two possible solutions for a quadratic equation.

$$x = \pm 5$$

Check to make sure that both solutions satisfy the equation.

Check: $x = -5$

$$\begin{aligned} \text{LS: } & \log_2(-5 + 3) + \log_2(-5 - 3) \\ & = \log_2(-2) + \log_2(-8) \\ & \neq \text{RS} \end{aligned}$$

When $x = -5$, the expression on the left side is undefined, since the logarithm of any negative number is undefined. Therefore, $x = -5$ is not a solution to the original equation. It is an inadmissible solution.

Check: $x = 5$

$$\begin{aligned} \text{LS: } & \log_2(5 + 3) + \log_2(5 - 3) \\ & = \log_2(8) + \log_2(2) \\ & = 3 + 1 \\ & = 4 \\ & = \text{RS} \end{aligned}$$

When $x = 5$, the expression on the left side gives the value on the right side. Therefore, $x = 5$ is the solution to the original equation.

The solution is $x = 5$.

In Summary

Key Ideas

- A logarithmic equation can be solved by expressing it in exponential form and solving the resulting exponential equation.
- If $\log_a M = \log_a N$, then $M = N$, where a, M, $N > 0$.

Need to Know

- A logarithmic equation can be solved by simplifying it using the laws of logarithms.
- When solving logarithmic equations, be sure to check for inadmissible solutions. A solution is inadmissible if its substitution in the original equation results in an undefined value. Remember that the **argument** and the base of a logarithm must both be positive.

CHECK Your Understanding

1. Solve.

a) $\log_2 x = 2\log_2 5$
b) $\log_3 x = 4\log_3 3$
c) $\log x = 3\log 2$
d) $\log(x - 5) = \log 10$
e) $\log_2 8 = x$
f) $\log_2 x = \frac{1}{2}\log_2 3$

2. Solve.

a) $\log_x 625 = 4$
b) $\log_x 6 = -\frac{1}{2}$
c) $\log_5(2x - 1) = 2$
d) $\log(5x - 2) = 3$
e) $\log_x 0.04 = -2$
f) $\log_5(2x - 4) = \log_5 36$

3. Given the formula from Example 1 for the magnitude of an earthquake, $R = \log\left(\frac{a}{T}\right) + B$, determine the value of a if $R = 6.3$, $B = 4.2$, and $T = 1.6$.

PRACTISING

4. Solve.

a) $\log_x 27 = \frac{3}{2}$
b) $\log_x 5 = 2$
c) $\log_3(3x + 2) = 3$
d) $\log x = 4$
e) $\log_{\frac{1}{3}} 27 = x$
f) $\log_{\frac{1}{2}} x = -2$

5. Solve.

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a) $\log_2 x + \log_2 3 = 3$
b) $\log 3 + \log x = 1$
c) $\log_5 2x + \frac{1}{2}\log_5 9 = 2$
d) $\log_4 x - \log_4 2 = 2$
e) $3\log x - \log 3 = 2\log 3$
f) $\log_3 4x + \log_3 5 - \log_3 2 = 4$

6. Solve $\log_6 x + \log_6(x - 5) = 2$. Check for inadmissible roots.

7. Solve.

a) $\log_7(x + 1) + \log_7(x - 5) = 1$
b) $\log_3(x - 2) + \log_3 x = 1$
c) $\log_6 x - \log_6(x - 1) = 1$
d) $\log(2x + 1) + \log(x - 1) = \log 9$
e) $\log(x + 2) + \log(x - 1) = 1$
f) $3\log_2 x - \log_2 x = 8$

8. Describe the strategy that you would use to solve each of the following equations. (Do not solve.)

a) $\log_9 x = \log_9 4 + \log_9 5$
b) $\log x - \log 2 = 3$
c) $\log x = 2\log 8$

9. The loudness, L, of a sound in decibels (dB) can be calculated using
A the formula $L = 10 \log \left(\frac{I}{I_0}\right)$, where I is the intensity of the sound in watts per square metre (W/m^2) and $I_0 = 10^{-12}\ \text{W/m}^2$.

a) A teacher is speaking to a class. Determine the intensity of the teacher's voice if the sound level is 50 dB.

b) Determine the intensity of the music in the earpiece of an MP3 player if the sound level is 84 dB.

10. Solve $\log_a(x + 2) + \log_a(x - 1) = \log_a(8 - 2x)$.
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11. Use graphing technology to solve each equation to two decimal places.

a) $\log (x + 3) = \log (7 - 4x)$ **c)** $2 \log x = 1$

b) $5^x = 3^{x+1}$ **d)** $\log (4x) = \log (x + 1)$

12. Solve $\log_5(x - 1) + \log_5(x - 2) - \log_5(x + 6) = 0$.

13. Explain why there are no solutions to the equations $\log_3(-8) = x$ and $\log_{-3}9 = x$.

14. **a)** Without solving the equation, state the restrictions on the variable x in the following: $\log (2x - 5) - \log (x - 3) = 5$

b) Why do these restrictions exist?

15. If $\log \left(\frac{x + y}{5}\right) = \frac{1}{2}(\log x + \log y)$, where $x > 0$, $y > 0$, show that $x^2 + y^2 = 23xy$.

16. Solve $\dfrac{\log (35 - x^3)}{\log (5 - x)} = 3$.

17. Given $\log_2 a + \log_2 b = 4$, calculate all the possible integer values
C of a and b. Explain your reasoning.

Extending

18. Solve the following system of equations algebraically.

$y = \log_2(5x + 4)$

$y = 3 + \log_2(x - 1)$

19. Solve each equation.

a) $\log_5(\log_3 x) = 0$ **b)** $\log_2(\log_4 x) = 1$

20. If $\left(\frac{1}{2}\right)^{x+y} = 16$ and $\log_{x-y}8 = -3$, calculate the values of x and y.