8.7 Solving Problems with Exponential and Logarithmic Functions

GOAL

Pose and solve problems based on applications of exponential and logarithmic functions.

YOU WILL NEED

- graphing calculator

INVESTIGATE the Math

The following data represent the prices of IBM personal computers and the demand for these computers at a computer store in 1997.

Price ($/computer)	2300	2000	1700	1500	1300	1200	1000
Demand (number of computers)	152	159	164	171	176	180	189

? Based on the data, what do you predict the demand would have been for computers priced at $1600?

A. What is the dependent variable in this situation? Enter the data into a graphing calculator, and create a scatter plot.

B. Is it clear what type of function you could use to model this situation? Explain.

C. Try fitting a function to the scatter plot you created. Try linear, quadratic, cubic, and exponential functions.

D. Use the regression feature of the calculator to determine the equation of the curve of best fit. Try linear, quadratic, cubic, and exponential regression.

E. Which type of function gives you the best fit?

F. Use the algebraic model you found to determine the price that would have a demand of 195 computers.

G. Use your model to predict the demand for computers priced at $1600.

Tech *Support*

For help using the graphing calculator to create scatter plots or using regression to determine the equation of best fit, see Technical Appendix, T-11.

Reflecting

H. How could you use the table of values to determine what type of function the data approximates?

I. How could you have used your graph to answer parts F and G?

APPLY the Math

EXAMPLE 1 Solving a problem using a logarithmic equation

In chemistry, the pH (the measure of acidity or alkalinity of a substance) is based on a logarithmic scale. A logarithmic scale uses powers of 10 to compare numbers that vary greatly in size. For example, very small and very large concentrations of the hydrogen ion in a solution influence its classification as either a base or an acid.

Concentration of hydrogen ions compared to distilled water		Examples of solutions at this pH
10 000 000	pH = 0	battery acid, strong hydrofluoric acid
1 000 000	pH = 1	hydrochloric acid secreted by stomach lining
100 000	pH = 2	lemon juice, gastric acid, vinegar
10 000	pH = 3	grapefruit, orange juice, soda
1000	pH = 4	tomato juice, acid rain
100	pH = 5	soft drinking water, black coffee
10	pH = 6	urine, saliva
1	pH = 7	"pure" water
$\frac{1}{10}$	pH = 8	seawater
$\frac{1}{100}$	pH = 9	baking soda
$\frac{1}{1000}$	pH = 10	Great Salt Lake, milk of magnesia
$\frac{1}{10\,000}$	pH = 11	ammonia solution
$\frac{1}{100\,000}$	pH = 12	soapy water
$\frac{1}{1\,000\,000}$	pH = 13	bleaches, oven cleaner
$\frac{1}{10\,000\,000}$	pH = 14	liquid drain cleaner

A difference of one pH unit represents a tenfold (10 times) change in the concentration of hydrogen ions in the solution. For example, the acidity of a sample with a pH of 5 is 10 times greater than the acidity of a sample with a pH of 6. A difference of 2 units, from 6 to 4, would mean that the acidity is 100 times greater, and so on.

- A liquid with a pH less than 7 is considered *acidic*.
- A liquid with a pH greater than 7 is considered *alkaline*.
- A liquid with a pH of 7 is considered *neutral*. Pure distilled water has a pH value of 7.

The relationship between pH and hydrogen ion concentration is given by the formula $\text{pH} = -\log[\text{H}^+]$, where $[\text{H}^+]$ is the concentration of hydrogen ions in moles per litre (mol/L).

a) Calculate the pH if the concentration of hydrogen ions is 0.0001 mol/L.

b) The pH of lemon juice is 2. Calculate the hydrogen ion concentration.

c) If the hydrogen ion concentration is a measure of the strength of an acid, how much stronger is an acid with pH 1.6 than an acid with pH 2.5?

Solution

a) $pH = -\log[H^+]$
$pH = -\log(0.0001)$
$pH = -(-4)$
$pH = 4$
The pH of the liquid is 4.

Substitute the value for $[H^+]$ into the equation. Evaluate log (0.0001).

b) $pH = -\log[H^+]$
$2 = -\log[H^+]$

Substitute the value 2 for the pH.

$-2 = \log[H^+]$

Divide both sides of the equation by -1.

$10^{-2} = [H^+]$

Rewrite the equation in exponential form.

$0.01 = [H^+]$

Evaluate the negative exponent to determine $[H^+]$.

The concentration of hydrogen ions is 0.01 mol/L.

c) $pH = -\log[H^+]$

$1.6 = -\log[H^+]$ $2.5 = -\log[H^+]$

To calculate the hydrogen ion concentration of both solutions, substitute the given pH values into the equation.

$10^{-1.6} = [H^+]$ $10^{-2.5} = [H^+]$
$0.0251 \doteq [H^+]$ $0.0032 \doteq [H^+]$

Express both equations in exponential form, and evaluate.

$$\frac{0.0251}{0.0032} = 7.84$$

Divide the concentration of the first acid by the concentration of the second acid to find the relative strength of the acids.

An acid with pH 1.6 is about 7.8 times stronger than an acid with pH 2.5.

EXAMPLE 2 Representing exponential values using the Richter scale

The Richter magnitude scale uses logarithms to compare intensity of earthquakes.

True Intensity	Richter Scale Magnitude
10^1	$\log_{10}10^1 = 1$
10^4	$\log_{10}10^4 = 4$
$10^{5.8}$	$\log_{10}10^{5.8} = 5.8$

An earthquake of magnitude 2 is actually 10 times more intense than an earthquake of magnitude 1. The difference between the magnitudes of two earthquakes can be used to determine the difference in intensity. If the average earthquake measures 4.5 on the Richter scale, how much more intense is an earthquake that measures 8?

Solution

$$\frac{10^8}{10^{4.5}} = 10^{8-4.5}$$

Calculate the quotient between the intensities.

$$= 10^{3.5}$$
$$\doteq 3162.3$$

Since the Richter scale is logarithmic, each step on the scale is a power of 10. The difference in intensity is calculated by evaluating 10 to the power of 3.5.

An earthquake that measures 8 on the Richter scale is about 3162 times more intense than an earthquake that measures 4.5.

Evaluate the power to compare the intensities of the two earthquakes.

EXAMPLE 3 Solving a problem using an exponential equation and logarithms

Blue jeans fade when washed due to the loss of blue dye from the fabric. If each washing removes about 2.2% of the original dye from the fabric, how many washings are required to give a pair of jeans a well-worn look? (For a well-worn look, jeans should contain, at most, 30% of the original dye.)

Solution

$$D(n) = (1 - 0.022)^n$$

Write an exponential model, using $D(n)$ to represent the percent of dye remaining as a decimal and n to represent the number of washings.

$$D(n) = (0.978)^n$$

Since the jeans are losing 2.2% of the dye each time, the ratio of decline is 0.978.

$$0.30 = (0.978)^n$$

Replace $D(n)$ with 0.30 since the well-worn look requires no more than 30% of the original dye remaining.

$$\log(0.30) = \log(0.978)^n$$

To solve for n, take the log of both sides of the equation.

$$\log(0.3) = n\log(0.978)$$

Rewrite the equation with the power as a coefficient.

$$\frac{\log(0.3)}{\log(0.978)} = n$$

Divide both sides of the equation by log (0.978) to solve for n.

$$54.12 \doteq n$$

It would take about 54 washings to give the jeans a well-worn look.

EXAMPLE 4 Solving a problem about sound intensity using logarithms

The dynamic range of human hearing and sound intensity spans from 10^{-12} W/m^2 to about 10 W/m^2. The highest sound intensity that can be heard is 10 000 000 000 000 times as loud as the quietest! This span of sound intensity is impractical for normal use. A more convenient way to express loudness is a relative logarithmic scale, with the lowest sound that can be heard by the human ear, $I_0 = 10^{-12}$ W/m^2, given the measure of loudness of 0 dB.

Recall that the formula that is used to measure sound is $L = 10\log\left(\frac{I}{I_0}\right)$, where L is the loudness measured in decibels, I is the intensity of the sound being measured, and I_0 is the intensity of sound at the threshold of hearing. The following table shows the loudness of a selection of sounds measured in decibels.

Sound	Loudness (dB)
soft whisper	30
normal conversation	60
shouting	80
subway	90
screaming	100
rock concert	120
jet engine	140
space-shuttle launch	180

How many times more intense is the sound of a rock concert than the sound of a subway?

Solution

$$L = 10\log\left(\frac{I}{I_0}\right)$$

$$120 = 10\log\left(\frac{I_{RC}}{I_0}\right) \qquad 90 = 10\log\left(\frac{I_S}{I_0}\right)$$

Let the intensity of sound for a rock concert be I_{RC} and for a subway be I_S. Find the values for the loudness of these sounds in the table, and substitute into the formula.

$$12 = \log\left(\frac{I_{RC}}{I_0}\right) \qquad 9 = \log\left(\frac{I_S}{I_0}\right)$$

Divide both sides of the equations by 10.

$$10^{12} = \frac{I_{RC}}{I_0} \qquad 10^9 = \frac{I_S}{I_0}$$

Express both equations in exponential form.

$$10^{12}I_0 = I_{RC} \qquad 10^9 I_0 = I_S$$

Isolate the variables for comparison.

$$\frac{I_{RC}}{I_S} = \frac{10^{12}I_0}{10^9 I_0} = 10^3 = 1000$$

Divide the results to compare the sound of a rock concert with the sound of a subway.

The sound of a rock concert is 1000 times more intense than the sound of a subway.

In Summary

Key Ideas

- When a range of values can vary greatly, using a logarithmic scale with powers of 10 makes comparisons between the large and small values more manageable.
- Growth and decay situations can be modelled by exponential functions of the form $f(x) = ab^x$. Note that
  - $f(x)$ is the final amount or number
  - a is the initial amount or number
  - for exponential growth, $b = 1 +$ growth rate
  - for exponential decay, $b = 1 -$ decay rate
  - x is the number of growth or decay periods

Need to Know

- Scales that measure a wide range of values, such as the pH scale, Richter scale, and decibel scale, are logarithmic scales.
- To compare concentrations on the pH scale, intensity on the Richter scale, or sound intensities, determine the quotient between the values being compared.
- Data from a table of values can be graphed and a curve of best fit determined. If the curve of best fit appears to be exponential, use the regression feature of the graphing calculator to determine an equation that models the data.

CHECK *Your Understanding*

1. If one earthquake has a magnitude of 5.2 on the Richter scale and a second earthquake has a magnitude of 6, compare the intensities of the two earthquakes.

2. Calculate the pH of a swimming pool with a hydrogen ion concentration of 6.21×10^{-8} mol/L.

3. A particular sound is 1 000 000 times more intense than a sound you can just barely hear. What is the loudness of the sound in decibels?

PRACTISING

4. **K** The loudness of a heavy snore is 69 dB. How many times as loud as a normal conversation of 60 dB is a heavy snore?

5. Calculate the hydrogen ion concentration of each substance.
   a) baking soda, with a pH of 9
   b) milk, with a pH of 6.6
   c) an egg, with a pH of 7.8
   d) oven cleaner, with a pH of 13

6. Calculate to two decimal places the pH of a solution with each concentration of H^+.
   a) concentration of $H^+ = 0.000\,32$
   b) concentration of $H^+ = 0.000\,3$
   c) concentration of $H^+ = 0.000\,045$
   d) concentration of $H^+ = 0.005$

7. a) Distilled water has an H^+ concentration of 10^{-7} mol/L. Calculate the pH of distilled water.
   b) Drinking water from a particular tap has a pH between 6.3 and 6.6. Is this tap water more or less acidic than distilled water? Explain your answer.

8. **A** The sound level of a moving power lawn mower is 109 dB. The noise level in front of the amplifiers at a concert is about 118 dB. How many times louder is the noise at the front of the amplifiers than the noise of a moving power lawn mower?

9. The following data represent the amount of an investment over 10 years.

Year	0	1	2	5	7	9	10
Amount ($)	5000	5321	5662.61	6824.74	7729.17	8753.45	9315.42

   a) Create a scatter plot, and determine the equation that models this situation.
   b) Determine the average annual interest rate.
   c) Use your equation to determine how long it took for the investment to double.

10. The intensity, I, of light passing through water can be modelled by the equation $I = 10^{1-0.13x}$, where x is the depth of the water in metres. Most aquatic plants require a light intensity of 4.2 units for strong growth. Determine the depth of water at which most aquatic plants receive the required light.

11. The following data represent the growth of a bacteria population over time.

Number of Hours	0	7	12	20	42
Number of Bacteria	850	2250	4500	13 500	287 200

   a) Create both a graphical model and an algebraic model for the data.
   b) Determine the length of time it took for the population to double.

12. The amount of water vapour in the air is a function of temperature, as shown in the following table.
T

Temperature (°C)	0	5	10	15	20	25	30	35
Saturation (mL/m^3 of air)	4.847	6.797	9.399	12.830	17.300	23.050	30.380	39.630

a) Calculate the growth factors for the saturation row of the table, to the nearest tenth.
b) Determine the average growth factor.
c) Write an exponential model for the amount of water vapour as a function of the temperature.
d) Determine the exponential function with a graphing calculator, using exponential regression.
e) What temperature change will double the amount of water in 1 m^3 of air?

13. Dry cleaners use a cleaning fluid that is purified by evaporation and condensation after each cleaning cycle. Every time the fluid is purified, 2.1% of it is lost. The fluid has to be topped up when half of the original fluid remains. After how many cycles will the fluid need to be topped up?

14. How long will it take for \$2500 to accumulate to \$4000 if it is invested at an interest rate of 6.5%/a, compounded annually?

15. A wound, initially with an area of 80 cm^2, heals according to the formula $A(t) = 80(10^{-0.023t})$, where $A(t)$ is the area of the wound in square centimetres after t days of healing. In how many days will 75% of the wound be healed?

16. Create a problem that could be solved using logarithms and another problem that could be solved without using logarithms. Explain how the two problems are different.
C

Extending

17. A new car has an interior sound level of 70 dB at 50 km/h. A second car, at the same speed, has an interior sound level that is two times more intense than that of the new car. Calculate the sound level inside the second car.

18. Assume that the annual rate of inflation will average 3.8% over the next 10 years.
a) Write an equation to model the approximate cost, C, of goods and services during any year in the next decade.
b) If the price of a brake job for a car is presently \$400, estimate the price 10 years from now.
c) If the price of an oil change 10 years from now will be \$47.95, estimate the price of an oil change today.