8.8 Rates of Change in Exponential and Logarithmic Functions

YOU WILL NEED

- graphing calculator

GOAL

Solve problems that involve average and instantaneous rates of change of exponential and logarithmic functions.

INVESTIGATE the Math

The following data from the U.S. Census Bureau represent the population of the United States, to the nearest million, every 10 years from 1900 to 2000.

Year	1900	1910	1920	1930	1940	1950	1960	1970	1980	1990	2000
Population (millions)	76	92	106	123	132	151	179	203	227	249	281

? At what rate was the population changing in the United States at the start of 1950?

A. Calculate the average rate of change in population over the entire 100 years.

B. Calculate the average rate of change in population over the first 50 years and over the last 50 years. Is the average rate of change in each 50-year period the same, less than, or greater than the average rate of change for the entire time period? Suggest reasons.

C. Estimate the instantaneous rate of change in population at the start of 1950 using an average rate of change calculation and a centred interval.

D. Use a graphing calculator to create a scatter plot using years since 1900 as the independent variable.

E. Determine an exponential equation that models the data.

F. Estimate the instantaneous rate of change in population at the start of 1950 using the model you found and a very small interval after 1950.

G. Estimate the instantaneous rate of change in population at the start of 1950 by drawing the appropriate tangent on your graph.

H. Compare your estimates from parts C, F, and G. Which estimate better represents the rate at which the U.S. population was changing in 1950? Explain.

Tech *Support*

For help using a graphing calculator to create scatter plots, and using regression to determine the equation of best fit, see Technical Appendix, T-11.

Reflecting

I. How could you use your graph to determine the year that had the least or greatest instantaneous change in population?

J. Describe how the rate at which the U.S. population grew changed during the period from 1900 to 2000.

APPLY the Math

EXAMPLE 1 Selecting a numerical strategy to calculate the average rate of change

The average number of students per computer in public schools is given in the table. Year 1 is 1983.

a) Calculate the average rate of change in students per computer during the entire time period and during the middle five years of the data.

b) What conclusions can you draw?

Year	Students per Computer
1	125
2	75
3	50
4	37
5	32
6	25
7	22
8	20
9	18
10	16
11	14
12	10.5
13	10

Solution

a) $$\text{Average rate of change} = \frac{10 - 125}{13 - 1}$$

To calculate the average rate of change during the entire time period, use the values for the number of students per computer for year 13 and year 1.

$$= \frac{-115}{12}$$

Average the difference over 12 years.

$$\doteq -9.58$$

The average rate of change in students per computer decreased by about 10 students per computer.

The middle five years are years 5 to 9.

$$\text{Average rate of change} = \frac{18 - 32}{9 - 5}$$

Calculate the difference in the number of students per computer for years 9 and 5.

$$= \frac{-14}{4}$$

Divide the difference by 4, since years 5 to 9 are a 4-year span.

$$= -3.5$$

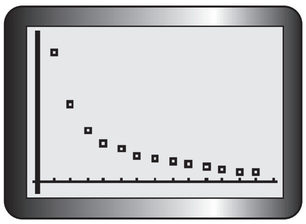
The average rate of change in students per computer decreased by 3.5 students per computer.

b) Since the rate of decline was faster over the entire period than during the middle period, the greatest change was either in the first four years or the last four years. The data show that there was a greater change in the number of students per computer during the first four years, so the decline was faster during this period.

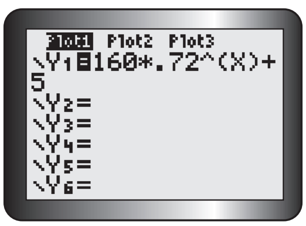
EXAMPLE 2 Selecting a strategy for calculating the instantaneous rate of change

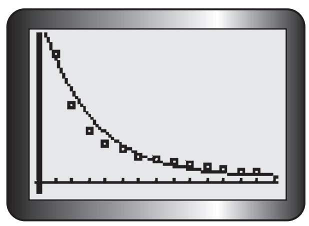
Using the data from Example 1, determine the instantaneous rate of change in students per computer for year 8.

Solution A: Calculating numerically

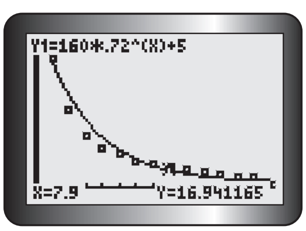


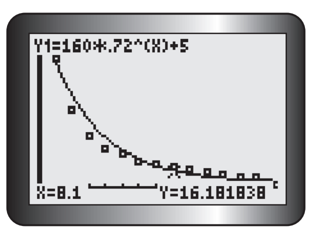
Plot the data on a graphing calculator.





Use an exponential function of the form $y = ab^x + c$ to fit a curve of the data.





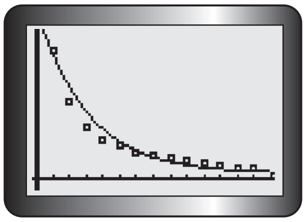
Use the VALUE feature to find values on either side of year 8.

$$\text{Instantaneous rate of change} = \frac{16.181\,838 - 16.941\,165}{8.1 - 7.9}$$
$$\doteq -3.4$$

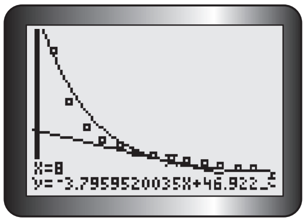
Use these values to calculate the slope of the tangent numerically.

At year 8, the instantaneous rate of change in students per computer was decreasing by about 3 students per computer.

Solution B: Calculating graphically



Plot the data, and then fit an exponential curve to the data.



Position the cursor at year 8, and draw the tangent.

The equation of the tangent is $y = -3.8x + 46.9$.

The calculator provides the equation of the tangent.

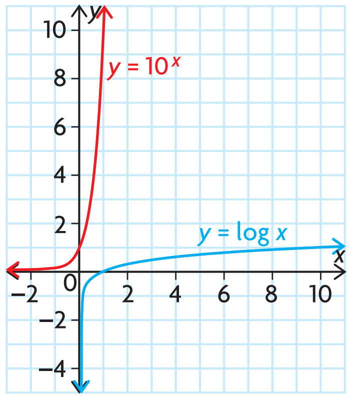
The slope of the tangent is -3.8.

At year 8, the instantaneous rate of change in students per computer decreased by about 4 students per computer.

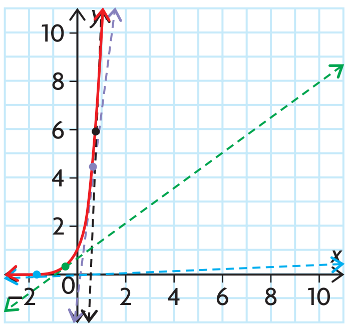
The slope of the tangent at year 8 gives the instantaneous rate of decline.

EXAMPLE 3 Comparing instantaneous rates of change in exponential and logarithmic functions

The graphs of $y = 10^x$ and $y = \log x$ are shown below. Discuss how the instantaneous rate of change in the y-values for each function changes as x grows larger.

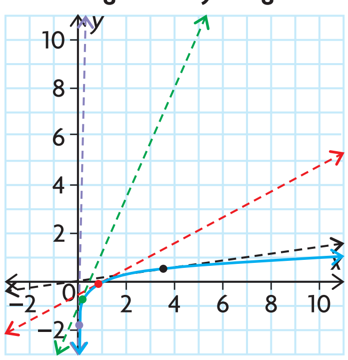


Solution



The graph of $y = 10^x$ has a horizontal asymptote. This means that the tangent lines at points with large negative values of x have a small slope, because the tangent lines are almost horizontal. As x increases, the slopes of the tangent lines also increase.

The instantaneous rate of change in the y-values is close to 0 for small values of x. As x increases, the instantaneous rate of change gets large very quickly. As $x \to \infty$, the instantaneous rate of change $\to \infty$.



The graph of $y = \log x$ has a vertical asymptote. This means that the tangent lines at points with very small values of x have very large slopes, because the tangent lines are almost vertical. As x gets larger, the tangent lines become less steep. When x is relatively small, small increases in x result in large changes in the slope of the tangent line. As x grows larger, however, the changes in the slope of the tangent line become smaller and the tangent slopes approach zero.

The instantaneous rate of change in the y-values is very large for small values of x. As x gets larger, the instantaneous rate of change gets smaller very quickly. As $x \to \infty$, the instantaneous rate of change $\to 0$.

In Summary

Key Ideas

- The average rate of change is not constant for exponential and logarithmic functions.
- The instantaneous rate of change at a particular point can be estimated by using the same strategies used with polynomial, rational, and trigonometric functions.

Need to Know

- The instantaneous rate of change for an exponential or logarithmic function can be determined numerically or graphically.
- The graph of an exponential or logarithmic function can be used to determine the period during which the average rate of change is least or greatest.
- The graph of an exponential or logarithmic function can be used to predict the greatest and least instantaneous rates of change and when they occur.

CHECK Your Understanding

Use the data from Example 1 for questions 1 to 3.

1. Calculate the average rate of change in number of students per computer during the following time periods.
a) years 2 to 10 **b)** years 1 to 5 **c)** years 10 to 13

2. Predict when the instantaneous rate of change in number of students per computer was the greatest. Give a reason for your answer.

3. Estimate the instantaneous rate of change in number of students per computer for the following years.
a) year 2 **b)** year 7 **c)** year 12

PRACTISING

4. Jerry invests \$6000 at 7.5%/a, compounded annually.
a) Determine the equation of the amount, A, after t years.
b) Estimate the instantaneous rate of change in the value at 10 years.
c) Suppose that the interest rate was compounded semi-annually instead of annually. What would the instantaneous rate of change be at 10 years?

5. **K** You invest \$1000 in a savings account that pays 6%/a, compounded annually.
a) Calculate the rate at which the amount is growing over the first
i) 2 years **ii)** 5 years **iii)** 10 years
b) Why is the rate of change not constant?

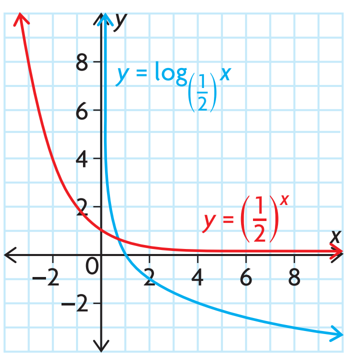
6. For 500 g of a radioactive substance with a half-life of 5.2 h, the amount remaining is given by the formula $M(t) = 500(0.5)^{\frac{t}{5.2}}$, where M is the mass remaining and t is the time in hours.
a) Calculate the amount remaining after 1 day.
b) Estimate the instantaneous rate of change in mass at 1 day.

7. The table shows how the mass of a chicken embryo inside an egg changes over the first 20 days after the egg is laid.
a) Calculate the average rate of change in the mass of the embryo from day 1 to day 20.
b) Determine an exponential equation that models the data.
c) Estimate the instantaneous rate of change in mass for the following days.
i) day 4 **ii)** day 12 **iii)** day 20
d) According to your model, when will the mass be 6.0000 g?

Days after Egg is Laid	Mass of Embryo (g)
1	0.0002
4	0.0500
8	1.1500
12	5.0700
16	15.9800
20	30.2100

8. A certain radioactive substance decays exponentially. The percent, P, of the substance left after t years is given by the formula $P(t) = 100(1.2)^{-t}$.
   a) Determine the half-life of the substance.
   b) Estimate the instantaneous rate of decay at the end of the first half-life period.

9. **A** The population of a town is decreasing at a rate of 1.8%/a. The current population of the town is 12 000.
   a) Write an equation that models the population of the town.
   b) Estimate the instantaneous rate of change in the population 10 years from now.
   c) Determine the instantaneous rate of change when the population is half its current population.



10. **T** The graphs of $y = \left(\frac{1}{2}\right)^x$ and $y = \log_{\frac{1}{2}} x$ are given. Discuss how the instantaneous rate of change for each function changes as x grows larger.

11. As a tornado moves, its speed increases. The function $S(d) = 93 \log d + 65$ relates the speed of the wind, S, in miles per hour, near the centre of a tornado to the distance that the tornado has travelled, d, in miles.
   a) Graph this function.
   b) Calculate the average rate of change for the speed of the wind at the centre of a tornado from mile 10 to mile 100.
   c) Estimate the rate at which the speed of the wind at the centre of a tornado is changing at the moment it has travelled its 10th mile and its 100th mile.
   d) Use your graph to discuss how the rate at which the speed of the wind at the centre of a tornado changes as the distance that the tornado travels increases.

12. **C** Explain how you could estimate the instantaneous rate of change for an exponential function if you did not have access to a graphing calculator.

Extending

13. How is the instantaneous rate of change affected by changes in the parameters of the function?
   a) $y = a \log[k(x - d)] + c$
   b) $y = ab^{[k(x-d)]} + c$