9.2 Combining Two Functions: Sums and Differences

GOAL

Represent the sums and differences of two functions graphically and algebraically, and determine their properties.

YOU WILL NEED

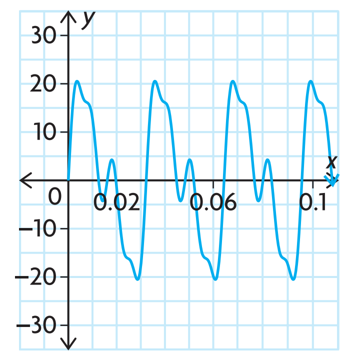
- graphing calculator or graphing software

INVESTIGATE the Math

The sound produced when a person strums a guitar chord represents the combination of sounds made by several different strings. The sound made by each string can be represented by a sine function. The period of each function is based on the frequency of the sound, whereas the loudness of the individual sounds varies and is related to the amplitude of each function. These sine functions are literally added together to produce the desired sound.

The sound of a G chord played on a six-string acoustic guitar can be approximated by the following combination of sine functions:

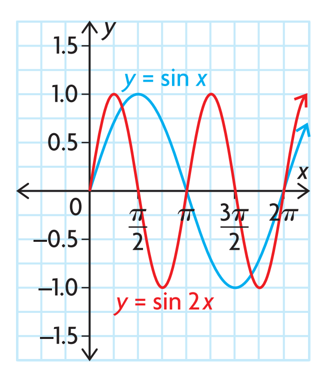
$y = 16 \sin 196x + 9 \sin 392x + 4 \sin 784x$



? When functions are added or subtracted, how do the resulting characteristics of the new function compare with those of the original functions?

A. Explore a similar but simpler combination of sine functions by examining the properties of the sum defined by $y = \sin x + \sin 2x$. Copy and complete the table of values, and use your results and the graphs shown to sketch the graph of $y = \sin x + \sin 2x$, where $0 \le x \le 2\pi$.

x	$\sin x$	$\sin 2x$	$\sin x + \sin 2x$
0	0	0	
$\frac{\pi}{4}$	0.7071	1	
$\frac{\pi}{2}$	1	0	
$\frac{3\pi}{4}$	0.7071	−1	
π	0	0	
$\frac{5\pi}{4}$	−0.7071	1	
$\frac{3\pi}{2}$	−1	0	
$\frac{7\pi}{4}$	−0.7071	−1	
2π	0	0	



B. Set the calculator to radian mode. Adjust the window settings so that $0 \le x \le 4\pi$ using an Xscl $= \frac{\pi}{4}$, and $-2 \le y \le 2$ using a Yscl $= 1$. Verify your graph in part A by graphing $y = \sin x + \sin 2x$.

C. What is the period of $y = \sin x + \sin 2x$? How does it compare with the periods of $y = \sin x$ and $y = \sin 2x$?

D. What is the amplitude of $y = \sin x + \sin 2x$? How does it compare with the amplitudes of $y = \sin x$ and $y = \sin 2x$?

E. Create a new table of values, and use your results and the graphs of $y = \sin x$ and $y = \sin 2x$ to sketch the graph of $y = \sin x - \sin 2x$, where $0 \le x \le 2\pi$. Repeat parts B to D using this difference function.

F. Do you think that the graph of $y = \sin 2x - \sin x$ will be the same as the graph you created in part E? Explain. Check your conjecture by using graphing technology to graph this function.

G. Investigate the sum of other types of functions. Use graphing technology to graph each set of functions, and describe how the characteristics of the functions are related.

i) $y_1 = -x, y_2 = x^2, y_3 = -x + x^2$
ii) $y_1 = \sqrt{x}, y_2 = \sqrt{x + 2}, y_3 = \sqrt{x} + \sqrt{x + 2}$
iii) $y_1 = 2^x,\ y_2 = 2^{-x}, y_3 = 2^x + 2^{-x}$
iv) $y_1 = \cos x, y_2 = \cos 2x, y_3 = \cos x + \cos 2x$

H. Investigate the difference of each set of functions in part G by graphing y_1 and y_2, and changing y_3 to $y_3 = y_1 - y_2$. Describe how the characteristics of the functions are related.

Reflecting

I. How does the degree of the sum or difference of two polynomial functions compare with the degree of the individual functions?

J. How does the period of the sum or difference of two trigonometric functions compare with the periods of the individual functions?

K. When looking at the sum of two functions, does the phrase "for each *x*, add the corresponding *y*-values together" describe the result you observed for every pair of functions? What phrase would you use to describe finding the difference of two functions?

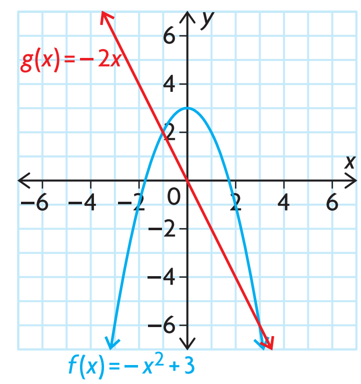
L. Looking at the graphs of the two square root functions, explain why the domain of the graph of their sum is $x \ge 0$.

M. Determine the y-intercept of y_3, where y_3 represents the difference of the two exponential functions. What does this point represent with respect to y_1 and y_2?

APPLY the Math

EXAMPLE 1 Selecting a strategy to combine functions by addition and subtraction

Given $f(x) = -x^2 + 3$ and $g(x) = -2x$, determine the graphs of $f(x) + g(x)$ and $f(x) - g(x)$. Discuss the key characteristics of the resulting graphs.

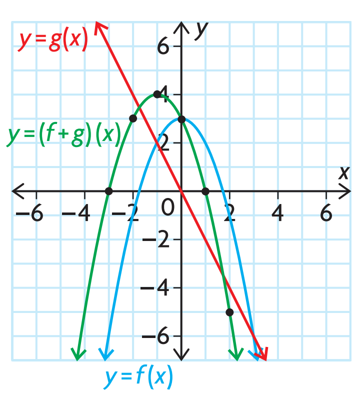


Solution A: Using a graphical strategy

x	$f(x)$	$g(x)$	$f(x) + g(x)$	$f(x) - g(x)$
-3	-6	6	$-6 + 6 = 0$	$-6 - 6 = -12$
-2	-1	4	3	-5
-1	2	2	4	0
-0.5	2.75	1	3.75	1.75
0	3	0	3	3
1	2	-2	0	4
2	-1	-4	-5	3
3	-6	-6	-12	0

Make a table of values for $f(x)$ and $g(x)$, for selected values of x. Create $f + g$ by adding the y-coordinates of f and g together. Create $f - g$ by subtracting the y-coordinates of g from f.

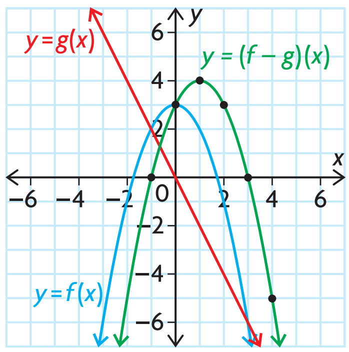
These functions can be added or subtracted over their entire domains since they both have the same domain $\{x \in \mathbf{R}\}$.



Plot the ordered pairs $(x, f(x) + g(x))$. Join the plotted points with a smooth curve.

Observe that the zeros of the new function occur when the y-values of f and g are the same distance from the x-axis, but on opposite sides. When a zero occurs for either f or g, the value of $f + g$ is the value of the other function.

At any point where f and g intersect, the value of $f + g$ is double the value of f (or g) for the corresponding x.



Plot the ordered pairs $(x, f(x) - g(x))$ from the table, and join them with a smooth curve to produce the graph of $f - g$.

Observe that the zeros of this $f - g$ graph occur when the graphs of f and g intersect.

Where g has a zero, the value of $f - g$ is the same as the value of f. Where f has a zero, the value of $f - g$ is the opposite of the value of g.

Solution B: Using an algebraic strategy

$f(x) = -x^2 + 3$ and $g(x) = -2x$

$$\begin{aligned}(f+g)(x) &= f(x) + g(x)\\ &= (-x^2 + 3) + (-2x)\\ &= -x^2 - 2x + 3\end{aligned}$$

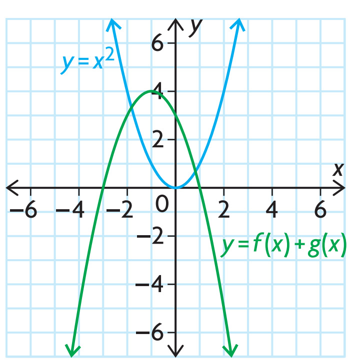
Remember that adding two functions means adding their y-values for a given value of x.

Since the expressions for $f(x)$ and $g(x)$ represent the y-values for each function, we determine an expression for $f + g$ by adding the two expressions.

$$\begin{aligned}(f+g)(x) &= -[x^2 + 2x] + 3\\ &= -[x^2 + 2x + 1 - 1] + 3\\ &= -[(x+1)^2 - 1] + 3\\ &= -(x+1)^2 + 4\end{aligned}$$

Recognizing that $f + g$ is a quadratic function, we can complete the square to change the expression into vertex form.

The graph of $f + g$ can be sketched by starting with the graph of $y = x^2$ and applying the following transformations: reflection in the x-axis, followed by a shift of 1 unit to the left and 4 units up.



The graph of $y = (f + g)(x)$ has the following characteristics: it is neither odd nor even; it is increasing on the interval $(-\infty, -1)$ and decreasing on the interval $(-1, \infty)$; it has zeros at $(-3, 0)$ and $(1, 0)$; it has a maximum value of $y = 4$ when $x = -1$; its domain is $\{x \in \mathbf{R}\}$; its range is $\{y \in \mathbf{R} \mid y \leq 4\}$.

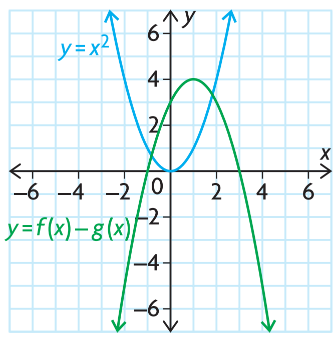
$$\begin{aligned}(f-g)(x) &= f(x) - g(x)\\ &= (-x^2 + 3) - (-2x)\\ &= -x^2 + 2x + 3\end{aligned}$$

Similarly, we obtain the expression for $f - g$ by subtracting $g(x)$ from $f(x)$.

In vertex form,

$$\begin{aligned}(f-g)(x) &= -[x^2 - 2x] + 3\\ &= -[x^2 - 2x + 1 - 1] + 3\\ &= -(x-1)^2 + 4\end{aligned}$$

Again, we can rewrite the quadratic expression in vertex form to graph it.



The graph of $f - g$ resembles the graph of $f + g$, except it has been shifted 1 unit to the right instead of 1 unit left.

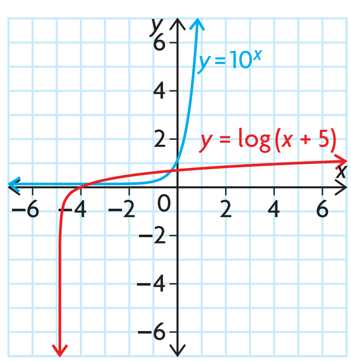
The graph of $y = (f - g)(x)$ has the following characteristics: it is neither odd nor even; it is increasing on the interval $(-\infty, 1)$ and decreasing on the interval $(1, \infty)$; it has zeros at $(-1, 0)$ and $(3, 0)$; it has a maximum value of $y = 4$ when $x = 1$; its domain is $\{x \in \mathbf{R}\}$; its range is $\{y \in \mathbf{R} | y \leq 4\}$.

EXAMPLE 2 Connecting the domains of the sum and difference of two functions

Determine the domain and range of $(f - g)(x)$ and $(f + g)(x)$ if $f(x) = 10^x$ and $g(x) = \log(x + 5)$.

Solution

Sketch the graphs of f and g.



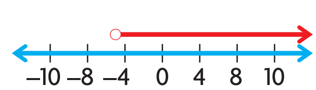
$f(x) = 10^x$ is an exponential function that has the x-axis as its horizontal asymptote. Exponential functions are defined for all real numbers, so its domain is $\{x \in \mathbf{R}\}$.

$g(x) = \log(x + 5)$ is a logarithmic function in base 10. Logarithmic functions are only defined for positive values: $x + 5 > 0$, so $x > -5$. This function has a vertical asymptote defined by $x = -5$. Its domain is $\{x \in \mathbf{R} | x > -5\}$.

$$
\begin{aligned}
(f - g)(x) &= f(x) - g(x) \\
&= 10^x - \log(x + 5) \\
(f + g)(x) &= f(x) + g(x) \\
&= 10^x + \log(x + 5)
\end{aligned}
$$

Values for the functions $f - g$ and $f + g$ can only be determined when functions f and g are both defined. This occurs for all values of x that are common to the domains of both f and g.

The domain of the functions $(f - g)(x)$ and $(f + g)(x)$ is $\{x \in \mathbf{R} | x > -5\}$.



This is the **intersection** of the domains of f and g.
$\{x \in \mathbf{R}\} \cap \{x \in \mathbf{R} | x > -5\}$
$= \{x \in \mathbf{R} | x > -5\}$

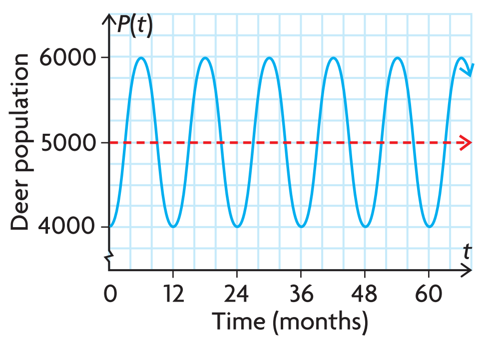
intersection
a set that contains the elements that are common to both sets; the symbol for intersection is $\cap$

EXAMPLE 3 Modelling a situation using a sum of two functions

In the past, biologists have found that the function $P(t) = 5000 - 1000 \cos\left(\frac{\pi}{6}t\right)$ models the deer population in a provincial park, which undergoes a seasonal fluctuation. In this case, $P(t)$ is the size of the deer population t months after January. A disease in the wolf population has caused its population to decline, and the biologists have discovered that the deer population is increasing by 50 deer each month. Assuming that this pattern continues, determine the new function that will model the deer population over time and discuss its characteristics.

Solution

Graph $P(t)$.



In the past, the deer population varied around its average yearly size of 5000. This is represented by the horizontal axis, or midline, of the graph shown.

This suggests that the function $P(t)$ can be thought of in the following way:

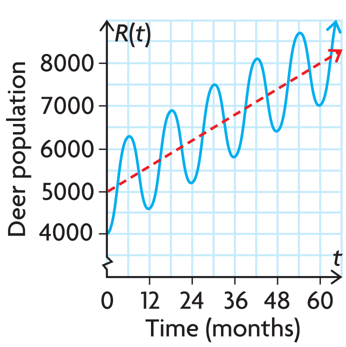
$$P(t) = \underbrace{5000}_{\text{average yearly population}} - \underbrace{1000 \cos\left(\frac{\pi}{6}t\right)}_{\text{seasonal variation}}$$

$$R(t) = \underbrace{5000 + 50t}_{\text{average yearly population increasing by 50 every month}} - \underbrace{1000 \cos\left(\frac{\pi}{6}t\right)}_{\text{seasonal variation}}$$

If the population of deer is increasing by 50 per month, then the average population could be represented by the expression $5000 + 50t$.

The new function $R(t)$ has been created by adding the function $f(t) = 50t$ to $P(t)$.

Therefore, $R(t) = P(t) + f(t)$.



The new population model has the following characteristics: it is neither odd nor even; it is increasing during the first six months of each year and decreasing during the last six months of each year; it has no zeros; it has no maximum or minimum value and its domain is $\{t \in \mathbf{R} \mid t \geq 0\}$; its range is $\{R(t) \in \mathbf{R} \mid R(t) \geq 4000\}$.

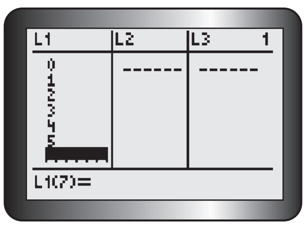
EXAMPLE 4 Reasoning about families of functions

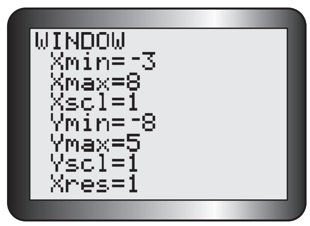
Use graphing technology to explore the graph of $f - g$, where $f(x) = x^2$ and $g(x) = nx$, and $n \in \mathbf{W}$. Discuss your results with respect to the type of function, its shape and symmetry, zeros, maximum and minimum values, intervals of increase/decrease, and domain and range.

Solution

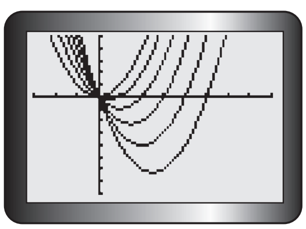
$(f - g)(x) = f(x) - g(x)$,
then for $f(x) = x^2$ and $g(x) = nx$,
$(f - g)(x) = x^2 - nx$, where $n \in \mathbf{W}$

$f - g$ will always be a quadratic function, regardless of the value of n.





Enter several values of $n \in \{0, 1, 2, 3, 4, 5\}$ into list 1 (L1) on a graphing calculator. Enter the equation $X^2 - L_1X$ into the equation editor. Then graph using the window settings shown.



The series of parabolas that $f - g$ produces will have identical shapes, since $a = 1$. It appears that, as n increases, the parabola is shifted to the right and down.

Most of these functions are neither odd nor even since their graphs are not symmetrical about the origin or y-axis.

The exception is when $n = 0$, which produces the even function $(f - g)(x) = x^2$.

The zeros of each parabola occur at $x = 0$ and $x = n$.

In factored form, $(f - g)(x) = x(x - n)$
$x = 0$ or $x - n = 0$
$x = n$

Since the parabola opens upward, the minimum value is $-\frac{n^2}{4}$, and the axis of symmetry is $x = \frac{n}{2}$.

The vertex of each parabola will occur at $\left(\frac{n}{2}, -\frac{n^2}{4}\right)$.

These functions are decreasing when $x \in \left(-\infty, \frac{n}{2}\right)$ and increasing when $x \in \left(\frac{n}{2}, \infty\right)$. The domain of $f - g$ is $\{x \in \mathbf{R}\}$, and the range is $y \in \left[-\frac{n^2}{4}, \infty\right)$.

Complete the square to put the function into vertex form.

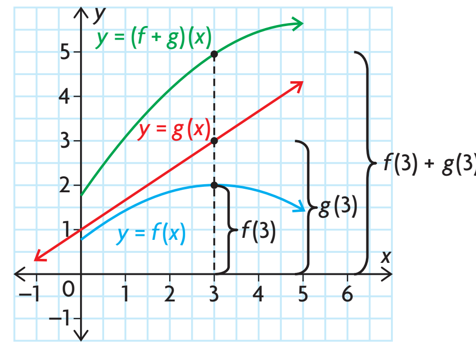
$$(f - g)(x) = x^2 - nx + \left(\frac{n}{2}\right)^2 - \left(\frac{n}{2}\right)^2$$

$$= \left(x - \frac{n}{2}\right)^2 - \frac{n^2}{4}$$

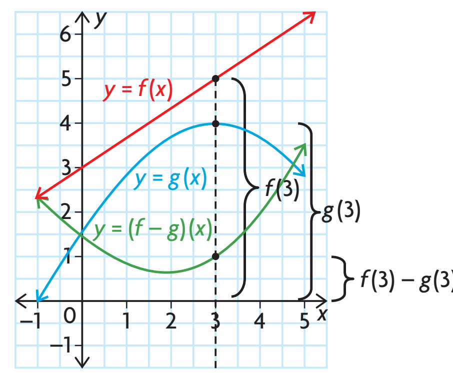
In Summary

Key Ideas

- When two functions $f(x)$ and $g(x)$ are combined to form the function $(f + g)(x)$, the new function is called the sum of f and g. For any given value of x, the value of the function is represented by $f(x) + g(x)$. The graph of $f + g$ can be obtained from the graphs of functions f and g by adding corresponding y-coordinates.



- Similarly, the difference of two functions, $f - g$, is $(f - g)(x) = f(x) - g(x)$. The graph of $f - g$ can be obtained by subtracting the y-coordinate of g from the y-coordinate of f for every pair of corresponding x-values.



Need to Know

- Algebraically, $(f + g)(x) = f(x) + g(x)$ and $(f - g)(x) = f(x) - g(x)$.
- The domain of $f + g$ or $f - g$ is the intersection of the domains of f and g. This means that the functions $f + g$ and $f - g$ are only defined where the domains of both f and g overlap.

CHECK Your Understanding

1. Let $f = \{(-4, 4), (-2, 4), (1, 3), (3, 5), (4, 6)\}$ and $g = \{(-4, 2), (-2, 1), (0, 2), (1, 2), (2, 2), (4, 4)\}$. Determine:

a) $f + g$ **c)** $f - g$ **e)** $f + f$

b) $g + f$ **d)** $g - f$ **f)** $g - g$

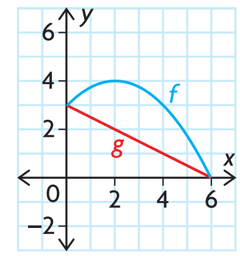
2. a) Determine $(f + g)(4)$ when $f(x) = x^2 - 3$ and $g(x) = -\frac{6}{x - 2}$.

b) For which value of x is $(f + g)(x)$ undefined? Explain why.

c) What is the domain of $(f + g)(x)$ and $(f - g)(x)$?

3. What is the domain of $f - g$, where $f(x) = \sqrt{x + 1}$ and $g(x) = 2\log[-(x + 1)]$?

4. Make a reasonable sketch of the graph of $f + g$ and $f - g$, where $0 \leq x \leq 6$, for the functions shown.



5. a) Given the function $f(x) = |x|$ (which is even) and $g(x) = x$ (which is odd), determine $f + g$.

b) Is $f + g$ even, odd, or neither?

PRACTISING

6. $f = \{(-9, -2), (-8, 5), (-6, 1), (-3, 7), (-1, -2), (0, -10)\}$
K and $g = \{(-7, 7), (-6, 6), (-5, 5), (-4, 4), (-3, 3)\}$.
Calculate:

a) $f + g$
b) $g + f$
c) $f - g$
d) $g - f$
e) $f - f$
f) $g + g$

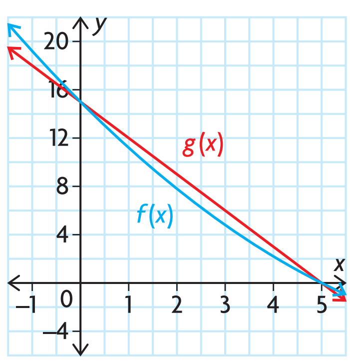
7. a) If $f(x) = \dfrac{1}{3x + 4}$ and $g(x) = \dfrac{1}{x - 2}$, what is $f + g$?

b) What is the domain of $f + g$?

c) What is $(f + g)(8)$?

d) What is $(f - g)(8)$?

8. The graphs of $f(x)$ and $g(x)$, where $0 \leq x \leq 5$, are shown. Sketch the graphs of $(f + g)(x)$ and $(f - g)(x)$.



9. For each pair of functions, determine the equations of $f(x) + g(x)$ and $f(x) - g(x)$. Using graphing technology, graph these new functions and discuss each of the following characteristics of the resulting graphs: symmetry, intervals of increase/decrease, zeros, maximum and minimum values, period (where applicable), and domain and range.

a) $f(x) = 2^x, g(x) = x^3$

b) $f(x) = \cos(2\pi x), g(x) = x^4$

c) $f(x) = \log(x), g(x) = 2x$

d) $f(x) = \sin(2\pi x), g(x) = 2\sin(\pi x)$

e) $f(x) = \sin(2\pi x) + 2, g(x) = \dfrac{1}{x}$

f) $f(x) = \sqrt{x - 2}, g(x) = \dfrac{1}{x - 2}$

10. **a)** Is the sum of two even functions even, odd, or neither? Explain.
b) Is the sum of two odd functions even, odd, or neither? Explain.
c) Is the sum of an even function and an odd function even, odd, or neither? Explain.

11. **A** Recall, from Example 3, the function $P(t) = 5000 - 1000 \cos\left(\frac{\pi}{6}t\right)$, which models the deer population in a provincial park. A disease in the deer population has caused it to decline. Biologists have discovered that the deer population is decreasing by 25 deer each month.
a) Assuming that this pattern continues, determine the new function that will model the deer population over time and discuss its characteristics.
b) Estimate when the deer population in this park will be extinct.

12. When the driver of a vehicle observes an obstacle in the vehicle's path, the driver reacts to apply the brakes and bring the vehicle to a complete stop. The distance that the vehicle travels while coming to a stop is a combination of the reaction distance, r, in metres, given by $r(x) = 0.21x$, and the braking distance, b, also in metres, given by $b(x) = 0.006x^2$. The speed of the vehicle is x km/h. Determine the stopping distance of the vehicle as a function of its speed, and calculate the stopping distance if the vehicle is travelling at 90 km/h.

13. **T** Determine a sine function, f, and a cosine function, g, such that $y = \sqrt{2}\sin(\pi(x - 2.25))$ can be written in the form of $f - g$.

14. Use graphing technology to explore the graph of $f + g$, where $f(x) = x^3$, $g(x) = nx^2$, and $n \in \mathbf{W}$. Discuss your results with respect to the type of function, its shape and symmetry, zeros, maximum and minimum values, intervals of increase/decrease, and domain and range.

15. **C** Describe or give an example of
a) two odd functions whose sum is an even function
b) two functions whose sum represents a vertical stretch applied to one of the functions
c) two rational functions whose difference is a constant function

Extending

16. Let $f(x) = x^2 - nx + 5$ and $g(x) = mx^2 + x - 3$. The functions are combined to form the new function $h(x) = f(x) + g(x)$. Points $(1, 3)$ and $(-2, 18)$ satisfy the new function. Determine the values of m and n.