9.3 Combining Two Functions: Products

GOAL

Represent the product of two functions graphically and algebraically, and determine the characteristics of the product.

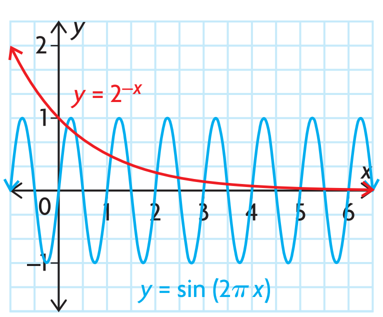
YOU WILL NEED

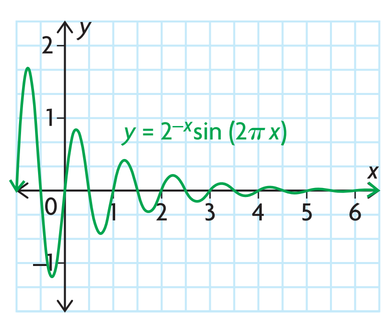
- graphing calculator or graphing software

LEARN ABOUT the Math

In the previous section, you learned that music is made up of combinations of sine waves. Have you ever wondered how sound engineers cause the music to fade out, gradually, at the end of a song? The music fades out because the sine waves that represent the music are being squashed or **damped**. Mathematically, this can be done by multiplying a sine function by another function.

The functions defined by $g(x) = \sin(2\pi x)$ and $f(x) = 2^{-x}$, where $\{x \in \mathbf{R} | x \geq 0\}$, are shown below. Observe what happens when these functions are multiplied to produce the graph of $(f \times g)(x) = 2^{-x} \sin(2\pi x)$.





? Can the product of two functions be constructed using the same strategies that are used to create the sum or difference of two functions?

EXAMPLE 1 Connecting the values of a product function to the values of each function

Investigate the product of the functions $f(x) = 2^{-x}$ and $g(x) = \sin(2\pi x)$.

Solution

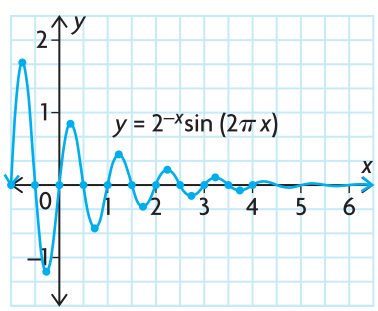
	A	B	C	D
1	x	f(x)=2^−x	g(x)=sin(2πx)	(fxg)(x)=(2^−x)sin(2πx)
2	0.00	1.00	0.00	0.00
3	0.25	0.84	1.00	0.84
4	0.50	0.71	0.00	0.00
5	0.75	0.59	–1.00	–0.59
6	1.00	0.50	0.00	0.00
7	1.25	0.42	1.00	0.42
8	1.50	0.35	0.00	0.00
9	1.75	0.30	–1.00	–0.30
10	2.00	0.25	0.00	0.00
11	2.25	0.21	1.00	0.21
12	2.50	0.18	0.00	0.00
13	2.75	0.15	–1.00	–0.15
14	3.00	0.13	0.00	0.00
15	3.25	0.11	1.00	0.11
16	3.50	0.09	0.00	0.00
17	3.75	0.07	–1.00	–0.07
18	4.00	0.06	0.00	0.00

In a spreadsheet, enter some values of x in column A, and enter the formulas for f, g, and $f \times g$ in columns B, C, and D, respectively.

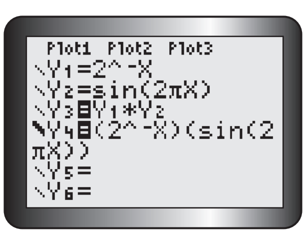
The values in the table have been rounded to two decimal places.

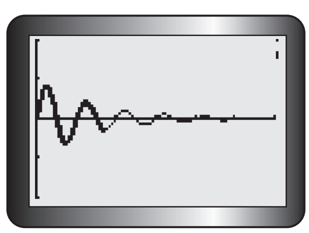
Looking at each row of the table, for any given value of x, the function value of $(f \times g)(x)$ is represented by $f(x) \times g(x)$.

This makes sense since the new function is created by multiplying the original functions together.

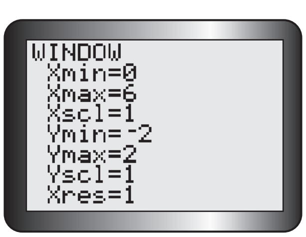


Plotting the ordered pairs $(x, (f \times g)(x))$ results in the graph of the dampened sine wave. This means that the graph of $f \times g$ can be obtained from the graphs of functions f and g by multiplying corresponding y-coordinates.





Use a graphing calculator to verify the results. Enter the functions into the equation editor as shown. Turn off the first two functions, and choose a bold line to graph the third function.



Use window settings that match the given graph of $(f \times g)(x)$.

The graph of Y4 traces over the graph of the product function Y3. This confirms that the product function is identical to, and obtained by, multiplying the expressions of the two functions together.

The graph of Y3 shows the graph produced by multiplying the corresponding *y*-values of the functions stored in Y1 and Y2.

Reflecting

A. If $(0.4, 0.76) \in f(x)$ and $(0.4, 0.59) \in g(x)$, what ordered pair belongs to $(f \times g)(x)$?

B. If $f(1) = 0.5$ and $(f \times g)(1) = 0$, what do you know about the value of $g(1)$? Explain.

C. Look at the original graphs of $f(x)$ and $g(x)$. How can you predict the locations of the zeros of $(f \times g)(x)$ before you construct a table of values or a graph? Explain.

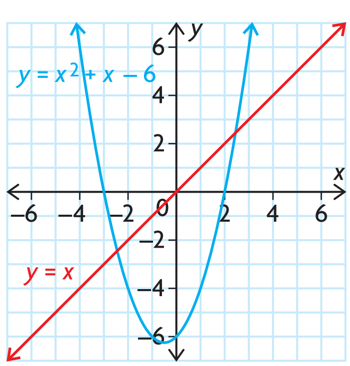
D. What is the domain of $f \times g$? How does it compare with the domains of f and g?

E. If function $f(x)$ was replaced by $f(x) = \sqrt{x}$, explain how this would change the domain of $(f \times g)(x)$.

APPLY the Math

EXAMPLE 2 Constructing the product of two functions graphically

Determine the graph of $y = (f \times g)(x)$, given the graphs of $f(x) = x^2 + x - 6$ and $g(x) = x$.



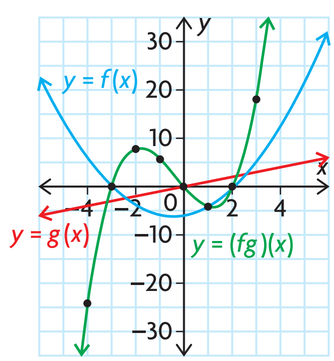
Solution

x	$f(x)$	$g(x)$	$(f \times g)(x)$
-4	6	-4	-24
-3	0	-3	0
-2	-4	-2	8
-1	-6	-1	6
0	-6	0	0
1	-4	1	-4
2	0	2	0
3	6	3	18
4	14	4	56

Use the graph to determine some of the points on the graphs of f and g, and create a table of values.

The graphs indicate that both functions have the same domain, $\{x \in \mathbf{R}\}$.

Determine the values of $(f \times g)(x)$ by multiplying the y-coordinates of f and g together for the same value of x.



The domain of the product function is the intersection of the domains of f and g, $\{x \in \mathbf{R}\}$.

Plot some of the ordered pairs $(x, (f \times g)(x))$, and use these to sketch the graph of the product function.

Notice that the zeros of the two functions, f and g, result in points that are also zeros of $f \times g$. This makes sense since the product of zero and any number is still zero.

Also notice that $(f \times g)(1) = f(1)$ because $g(1) = 1$. As a result, $(f \times g)(1) = f(1) \times 1 = -4 \times 1 = -4$. Similarly, $(f \times g)(-1) = -f(-1)$ because $g(-1) = -1$, so $(f \times g)(-1) = f(-1) \times (-1)$ $= -6 \times -1 = 6$.

Functions f and g are second and first degree polynomial functions, so the product function fg is a third degree polynomial function (also called a cubic function).

EXAMPLE 3 Constructing the product of two functions algebraically

Let $f(x) = \sqrt{x}$ and $g(x) = \frac{1}{2}x - 2$.

a) Find the equation of the function $(f \times g)(x)$.
b) Determine $(f \times g)(4)$.
c) Find the domain of $y = (f \times g)(x)$.
d) Use graphing technology to graph $y = (f \times g)(x)$, and discuss the key characteristics of the graph.

Solution

a) $(f \times g)(x) = f(x) \times g(x)$
$= \sqrt{x}\left(\frac{1}{2}x - 2\right)$

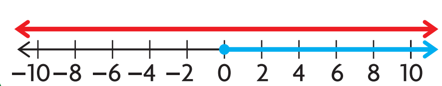
To find the formula for the product of the functions, take the expression for $f(x)$ and multiply it by the expression for $g(x)$.

b) $(f \times g)(4) = \sqrt{4}\left(\frac{1}{2}(4) - 2\right)$
$= 2(0)$
$= 0$

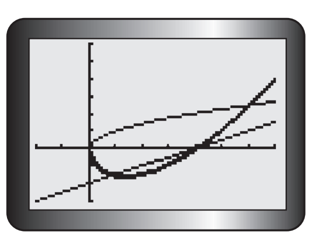
Calculate the value of $(f \times g)(4)$ by substituting $x = 4$ into the expression $(f \times g)(x)$.

c) The domain of g is $\{x \in \mathbf{R}\}$, but the domain of f is $\{x \in \mathbf{R} \mid x \geq 0\}$. So, the domain of $f \times g$ is $\{x \in \mathbf{R} \mid x \geq 0\}$.

The domain of $f \times g$ can only consist of x-values that exist in the domains of both f and g.



d)



The graph of $f \times g$ is the bold line.

The graph of $f \times g$
- lies below the x-axis when $x \in (0, 4)$, since $f(x) > 0$ and $g(x) < 0$ in that interval
- has zeros occurring at $x = 0$ when $f(x) = 0$ and at $x = 4$ when $g(x) = 0$; no other zeros will occur, since both functions are positive
- is neither odd nor even since it has no symmetry about the origin or the y-axis

EXAMPLE 4 Modelling a situation using a product function

The rate at which a contaminant leaves a storm sewer and enters a lake depends on two factors: the concentration of the contaminant in the water from the sewer and the rate at which the water leaves the sewer. Both of these factors vary with time. The concentration of the contaminant, in kilograms per cubic metre of water, is given by $c(t) = t^2$, where t is in seconds. The rate at which water leaves the sewer, in cubic metres per second, is given by $w(t) = \frac{1}{t^4 + 20}$. Determine the time at which the contaminant leaves the sewer and enters the lake at the maximum rate.

Solution

$c(t)$ is in $\frac{\text{kg}}{\text{m}^3}$ and $w(t)$ is in $\frac{\text{m}^3}{\text{s}}$

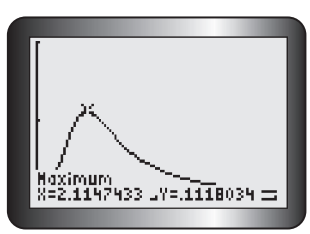
$$c(t) \times w(t) \rightarrow \left(\frac{\text{kg}}{\cancel{\text{m}^3}}\right)\left(\frac{\cancel{\text{m}^3}}{\text{s}}\right) = \frac{\text{kg}}{\text{s}}$$

The product of the concentration function and the water rate function results in a function that describes the rate of contaminant flow into the lake.

Analyze the units of both functions to help you determine the relationship between the functions that can be used to determine a function for the rate at which the contaminant flows into the lake.

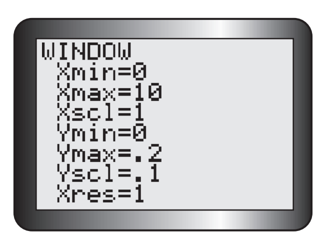
$$c(t) \times w(t) = (t^2)\left(\frac{1}{t^4 + 20}\right)$$
$$= \frac{t^2}{t^4 + 20}$$

In this context, the domain of both functions is $\{t \in \mathbf{R} \mid t \geq 0\}$ since both functions have time as the independent variable. Thus, $\{t \in \mathbf{R} \mid t \geq 0\}$ is also the domain of $c(t) \times w(t)$.



The contaminant is flowing into the lake at a maximum rate of about 0.11 kg/s. This occurs at about 2 s after the water begins to flow into the lake.

Use the maximum operation on a graphing calculator to graph $c(t) \times w(t)$ on its domain and estimate when its maximum value occurs.



Tech *Support*

For help determining the maximum value of a function using a graphing calculator, see Technical Appendix, T-9.

In Summary

Key Idea

- When two functions, $f(x)$ and $g(x)$, are combined to form the function $(f \times g)(x)$, the new function is called the product of f and g. For any given value of x, the function value is represented by $f(x) \times g(x)$. The graph of $f \times g$ can be obtained from the graphs of functions f and g by multiplying each y-coordinate of f by the corresponding y-coordinate of g.

Need to Know

- Algebraically, $f \times g$ is defined as $(f \times g)(x) = f(x) \cdot g(x)$.
- The domain of $f \times g$ is the intersection of the domains of f and g.
- If $f(x) = 0$ or $g(x) = 0$, then $(f \times g)(x) = 0$.
- If $f(x) = \pm 1$, then $(f \times g)(x) = \pm g(x)$. Similarly, if $g(x) = \pm 1$, then $(f \times g)(x) = \pm f(x)$.

CHECK Your Understanding

1. For each of the following pairs of functions, determine $(f \times g)(x)$.

a) $f(x) = \{(0, 2), (1, 5), (2, 7), (3, 12)\}$, $g(x) = \{(0, -1), (1, -2), (2, 3), (3, 5)\}$

b) $f(x) = \{(0, 3), (1, 6), (2, 10), (3, -5)\}$, $g(x) = \{(0, 4), (2, -2), (4, 1), (6, 3)\}$

c) $f(x) = x, g(x) = 4$

d) $f(x) = x, g(x) = 2x$

e) $f(x) = x + 2, g(x) = x^2 - 2x + 1$

f) $f(x) = 2^x, g(x) = \sqrt{x - 2}$

2. **a)** Graph each pair of functions in question 1, parts c) to f), on the same grid.

b) State the domains of f and g.

c) Use your graph to make an accurate sketch of $y = (f \times g)(x)$.

d) State the domain of $f \times g$.

3. If $f(x) = \sqrt{1 + x}$ and $g(x) = \sqrt{1 - x}$, determine the domain of $y = (f \times g)(x)$.

PRACTISING

4. Determine $(f \times g)(x)$ for each of the following pairs of functions.

K **a)** $f(x) = x - 7, g(x) = x + 7$

b) $f(x) = \sqrt{x + 10}, g(x) = \sqrt{x + 10}$

c) $f(x) = 7x^2, g(x) = x - 9$

d) $f(x) = -4x - 7, g(x) = 4x + 7$

e) $f(x) = 2 \sin x, g(x) = \dfrac{1}{x - 1}$

f) $f(x) = \log (x + 4), g(x) = 2^x$

5. For each of the problems in question 4, state the domain and range of $(f \times g)(x)$.

6. For each of the problems in question 4, use graphing technology to graph $(f \times g)(x)$ and then discuss each of the following characteristics of the graphs: symmetry, intervals of increase/decrease, zeros, maximum and minimum values, and period (where applicable).

7. The graph of the function $f(x)$ is a line passing through the origin with a slope of -4, whereas the graph of the function $g(x)$ is a line with a y-intercept of 1 and a slope of 6. Sketch the graph of $(f \times g)(x)$.

8. For each of the following pairs of functions, state the domain of $(f \times g)(x)$.

a) $f(x) = \frac{1}{x^2 - 5x - 14}, g(x) = \sec x$

b) $f(x) = 99^x, g(x) = \log(x - 8)$

c) $f(x) = \sqrt{x + 81}, g(x) = \csc x$

d) $f(x) = \log(x^2 + 6x + 9), g(x) = \sqrt{x^2 - 1}$

9. If the function $f(t)$ describes the per capita energy consumption in a particular country at time t, and the function $p(t)$ describes the population of the country at time t, then explain what the product function $(f \times p)(t)$ represents.

10. A An average of 20 000 people visit the Lakeside Amusement Park each day in the summer. The admission fee is \$25.00. Consultants predict that, for each \$1.00 increase in the admission fee, the park will lose an average of 750 customers each day.

a) Determine the function that represents the projected daily revenue if the admission fee is increased.

b) Is the revenue function a product function? Explain.

c) Estimate the ticket price that will maximize revenue.

11. A water purification company has patented a unique process to remove contaminants from a container of water at the same time that more contaminated water is added for purification. The percent of contaminated material in the container of water being purified can be modelled by the function $c(t) = (0.9)^t$, where t is the time in seconds. The number of litres of water in the container can be modelled by the function $l(t) = 650 + 300t$. Write a function that represents the number of litres of contaminated material in the container at any time t, and estimate when the amount of contaminated material is at its greatest.

12. Is the following statement true or false? "If $f(x) \times g(x)$ is an odd function, then both $f(x)$ and $g(x)$ are odd functions." Justify your answer.
T

13. Let $f(x) = mx^2 + 2x + 5$ and $g(x) = 2x^2 - nx - 2$. The functions are combined to form the new function $h(x) = f(x) \times g(x)$. Points $(1, -40)$ and $(-1, 24)$ satisfy the new function. Determine $f(x)$ and $g(x)$.

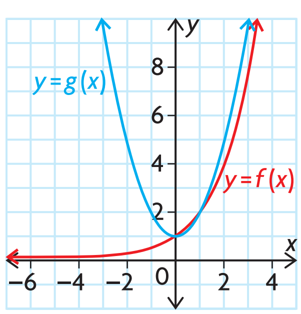
14. Let $f(x) = \sqrt{-x}$ and $g(x) = \log(x + 10)$.
C
- **a)** Determine the equation of the function $y = (f \times g)(x)$, and state its domain.
- **b)** Provide two different strategies for sketching $y = (f \times g)(x)$. Discuss the merits of each strategy.
- **c)** Choose one of the strategies you discussed in part b), and make an accurate sketch.

15.
- **a)** If $f(x) = x^2 - 25$, determine the equation of the product function $f(x) \times \frac{1}{f(x)}$.
- **b)** Determine the domain, and sketch the graph of the product function you found in part a).
- **c)** If $f(x)$ is a polynomial function, explain how the domain and range of $f(x) \times \frac{1}{f(x)}$ changes as the degree of $f(x)$ changes.

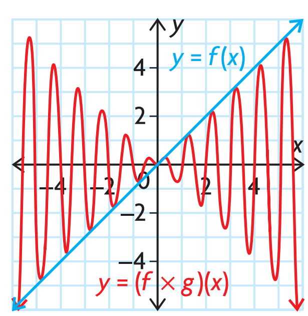
Extending

16. Given the following graphs, determine the equations of $y = f(x)$, $y = g(x)$, and $y = (f \times g)(x)$.

a)



b)



17. Determine two functions, f and g, whose product would result in each of the following functions.
- **a)** $(f \times g)(x) = 4x^2 - 81$
- **b)** $(f \times g)(x) = 8\sin^3 x + 27$
- **c)** $(f \times g)(x) = 4x^{\frac{5}{2}} - 3x^{\frac{3}{2}} + x^{\frac{1}{2}}$
- **d)** $(f \times g)(x) = \dfrac{6x - 5}{2x + 1}$