9.4 Exploring Quotients of Functions

YOU WILL NEED

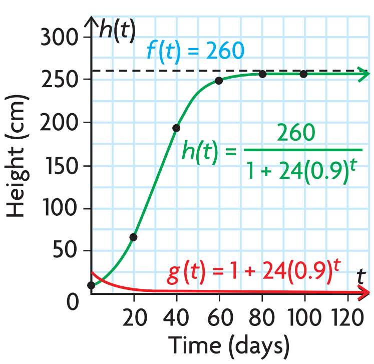
- graph paper
- graphing calculator or graphing software

GOAL

Represent the quotient of two functions graphically and algebraically, and determine the characteristics of the quotient.

EXPLORE the Math

The logistic function is often used to model growth. This function has the general equation $P(t) = \frac{c}{1 + ab^t}$, where $a > 0$, $0 < b < 1$, and $c > 0$. In this function, t is time. For example, the height of a sunflower plant can be modelled using the function $h(t) = \frac{260}{1 + 24(0.9)^t}$, where $h(t)$ is the height in centimetres and t is the time in days. The function $h(t) = \frac{f(t)}{g(t)}$ is the quotient of two functions, where $f(t) = 260$ (a constant function) and $g(t) = 1 + 24(0.9)^t$ (an exponential function). The table and graphs show that the values of a quotient function can be determined by dividing the values of the two functions.



t (days)	$f(t) = 260$	$g(t) = 1 + 24(0.9)^t$	$h(t) = \frac{260}{1 + 24(0.9)^t}$
0	260	25	$\frac{260}{25} = 10.4$
20	260	3.92	66.3
40	260	1.35	192.6
60	260	1.04	250.0
80	260	1.01	257.4
100	260	1.00	260.0

This function shows slow growth for small values of t, then rapid growth, and then slow growth again when the height of the sunflower approaches its maximum height of 260 cm.

The logistic function is an example of a quotient function. In function notation, we can express this as $(f \div g)(x) = f(x) \div g(x)$.

? What are the characteristics of functions that are produced by quotients of other types of functions?

A. Consider the function defined by $y = \frac{4}{x + 2}$ in the form $y = \frac{f(x)}{g(x)}$. Write the expressions for functions f and g.

B. On graph paper, draw and label the graphs of $y = f(x)$ and $y = g(x)$, and state their domains.

C. Locate any points on your graph of g where $g(x) = 0$. What will happen when you calculate the value of $f \div g$ for these x-coordinates? How would this appear on a graph?

D. Locate any points on your graph where $g(x) = \pm 1$. What values of x produced these results? Explain how you could determine these x-values algebraically.

E. Determine the value of $f \div g$ for each of the x's in part D. How do your answers compare with the corresponding values of f? Explain.

F. Over what interval(s) is $g(x) > 0$? Over what interval(s) is $f(x) > 0$?

G. Determine all the intervals where both f and g are positive or where both are negative. Will the function $f \div g$ be positive in the same intervals? Justify your answer.

H. Determine any intervals where either f or g is positive and the other is negative. Discuss the behaviour of $f \div g$ over these intervals. If no such intervals exist, what implication would this have for $f \div g$? Explain.

I. For what values of x is $(f \div g)(x) = f(x)$? For what values of x is $(f \div g)(x) = -f(x)$?

J. Using all the information about $f \div g$ that you have determined, make an accurate sketch of $y = (f \div g)(x)$ and state its domain.

K. Verify your results by graphing f, g, and $f \div g$ using graphing technology.

L. Repeat parts A to K using the following functions.

i) $y = \dfrac{x + 1}{(x + 3)(x - 1)}$

ii) $y = \dfrac{4}{x^2 + 1}$

iii) $y = \dfrac{\sin x}{x}$

iv) $y = \dfrac{2^x}{\sqrt{x}}$

Reflecting

M. The graphs of $y = \dfrac{4}{x + 2}$, $y = \dfrac{x + 1}{(x + 3)(x - 1)}$, and $y = \dfrac{2x}{\sqrt{x}}$ have vertical asymptotes, but the graphs of $h(t) = \dfrac{260}{1 + 24(0.9)^t}$, $y = \dfrac{4}{x^2 + 1}$, and $y = \dfrac{\sin x}{x}$ do not. Explain.

N. The graph of $y = \dfrac{x + 1}{(x + 3)(x - 1)}$ lies above the x-axis in the interval $x \in (-3, -1)$. By examining the behaviour of functions f and g, explain how you can reach this conclusion.

In Summary

Key Idea

- When two functions, $f(x)$ and $g(x)$, are combined to form the function $(f \div g)(x)$, the new function is called the quotient of f and g. For any given value of x, the value of the function is represented by $f(x) \div g(x)$. The graph of $f \div g$ can be obtained from the graphs of functions f and g by dividing each y-coordinate of f by the corresponding y-coordinate of g.

Need to Know

- Algebraically, $(f \div g)(x) = f(x) \div g(x)$.
- $f \div g$ will be defined for all x-values that are in the intersection of the domains of f and g, except in the case where $g(x) = 0$. If the domain of f is A, and the domain of g is B, then the domain of $f \div g$ is $\{x \in \mathbf{R} | x \in A \cap B, g(x) \neq 0\}$.
- If $f(x) = 0$ when $g(x) \neq 0$, then $(f \div g)(x) = 0$.
- If $f(x) = \pm 1$, then $(f \div g)(x) = \pm \frac{1}{g(x)}$. Similarly, if $g(x) = \pm 1$, then $(f \div g)(x) = \pm f(x)$. Also, if $f(x) = \pm g(x)$, then $(f \div g)(x) = \pm 1$

Further Your Understanding

1. For each of the following pairs of functions, write the equation of $y = (f \div g)(x)$.

a) $f(x) = 5, g(x) = x$

b) $f(x) = 4x, g(x) = 2x - 1$

c) $f(x) = 4x, g(x) = x^2 + 4$

d) $f(x) = x + 2, g(x) = \sqrt{x - 2}$

e) $f(x) = 8, g(x) = 1 + \left(\frac{1}{2}\right)^x$

f) $f(x) = x^2, g(x) = \log(x)$

2. **a)** Graph each pair of functions in question 1 on the same grid.

b) State the domains of f and g.

c) Use your graphs to make an accurate sketch of $y = (f \div g)(x)$.

d) State the domain of $f \div g$.

3. Recall that the function $h(t) = \frac{260}{1 + 24(0.9)^t}$ models the growth of a sunflower, where $h(t)$ is the height in centimetres and t is the time in days.

a) Calculate the average rate of growth of the sunflower over the first 20 days.

b) Determine when the sunflower has grown to half of its maximum height.

c) Estimate the instantaneous rate of change in height at the time you found in part b).

d) What happens to the instantaneous rate of change in height as the sunflower approaches its maximum height? How does this relate to the shape of the graph?