9.5 Composition of Functions

GOAL

Determine the composition of two functions numerically, graphically, and algebraically.

LEARN ABOUT the Math

Sometimes you will find a situation in which two related functions are present. Often both functions are needed to analyze the situation or solve a problem.

Forest fires often spread in a roughly circular pattern. The area burned depends on the radius of the fire. The radius, in turn, may increase at a constant rate each day.

Suppose that $A(r) = \pi r^2$ represents the area, A, of a fire as a function of its radius, r. If the radius of the fire increases by 0.5 km/day, then $r(t) = 0.5t$ represents the radius of the fire as a function of time, t. The area is measured in square kilometres, the radius is measured in kilometres, and the time is measured in days.

? How can the area burned be determined on the sixth day of the fire?

EXAMPLE 1 Reasoning numerically, graphically, and algebraically about a composition of functions

Determine the area burned by the fire on the sixth day.

Solution A: Using graphical and numerical analysis

Use the given functions to make tables of values.

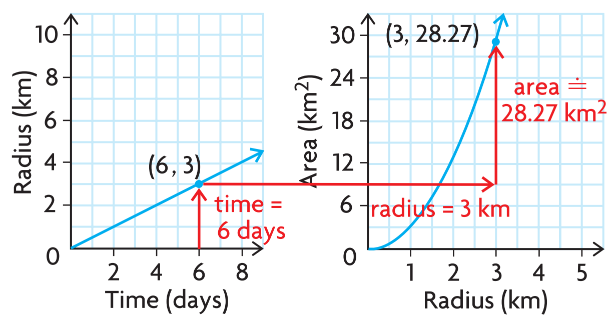
t	$r(t) = 0.5t$
0	0
2	1
4	2
6	3
8	4

r	$A(r) = \pi r^2$
0	0
1	3.14
2	12.57
3	28.27
4	50.27

Both time and radius must be positive, so $t \geq 0$ and $r \geq 0$.

$r(t)$ is a linear function, and $A(r)$ is a quadratic function.

Use the tables of values to sketch the graphs.



To find the radius of the area burned by the forest fire, the length of time that the fire has been burning must be known. Once the radius is known, the total area burned can be determined.

Reading from the first graph, the radius is 3 km when $t = 6$ days. Then reading from the second graph, a radius of 3 km indicates an area of about 28.3 km^2.

In the tables of values, time corresponds with radius, and radius corresponds with area.

$\boldsymbol{r}$: time $\rightarrow$ radius
$\boldsymbol{A}$: radius $\rightarrow$ area

The output in the first table becomes the input in the second table.

$$6 \xrightarrow{\quad} 3 \xrightarrow{\quad} 28.3$$

$$r(6) \qquad \begin{aligned} &A(r(6)) \\ &= A(3) \\ &\doteq 28.3 \end{aligned}$$

Determine the radius after six days, $r(6)$, and use it as the input for the area function, $A(r(6))$, to determine the area burned after six days.

$r(6) = 0.5(6) = 3$ and $A(3) = \pi(3)^2 \doteq 28.3$

The fire has burned about 28.3 km^2 on the sixth day.

Solution B: Using algebraic analysis

$$r = g(t) = 0.5t$$
$$A = f(r) = \pi r^2$$

The radius of the fire, r, grows at 0.5 km per day, so it is a function of time.

The area, A, of the fire increases in a circular pattern as its radius, r, increases, so it is a function of the circle's radius.

Since $r = g(t)$

$$A = f(r) = f(g(t))$$

To solve the problem, combine the area function with the radius function by using the output for the radius function as the input for the area function.

$$A = f(0.5t) \\ = \pi(0.5t)^2 \\ = 0.25\pi t^2$$

This process yields the composite function for the area of the fire as a function of time.

$$A(6) = 0.25\pi(6)^2 \\ = 0.25\pi(36) \\ = 9\pi \\ \doteq 28.3$$

To determine the size of the burn area after six days, substitute $t = 6$ into this new function.

The fire has burned an area of about 28.3 km^2 after six days.

composite function
a function that is the composite of two other functions; the function $f(g(t))$ is called the composition of f with g; the function $f(g(t))$ is denoted by $(f \circ g)(t)$ and is defined by using the output of the function g as the input for the function f

Reflecting

A. A point on the second graph was used to solve the problem. Explain how the x-coordinate of this point was determined.

B. What connection was observed between the tables of values for the two functions? Why does it make sense that there is a function that combines the two functions to solve the forest fire problem?

C. Explain how the two functions were combined algebraically to determine a single function that predicts the area burned for a given time. How is the range of r related to the domain of A in this combination?

Communication *Tip*
$f \circ g$ is read as "f operates on g" while $f(g(x))$ is read as "f of g of x."

APPLY the Math

EXAMPLE 2 Reasoning about the order in which two functions are composed

Given the functions $f(x) = 2x + 3$ and $g(x) = \sqrt{x}$, determine whether $(f \circ g)(x) = (g \circ f)(x)$.

Solution

$$(f \circ g)(x) = f(\underbrace{g(x)})$$

outer function

inner function

When f is composed with g, take the output for the inner function g and use it as the input for the outer function f.

$$x \xrightarrow{g} g(x) \xrightarrow{f} f(g(x))$$
$$g(x) = \sqrt{x}$$
$$\begin{aligned} f(g(x)) &= f(\sqrt{x}) \\ &= 2(\sqrt{x}) + 3 \\ &= 2\sqrt{x} + 3 \end{aligned}$$

The output for g is the expression $\sqrt{x}$. Use this as the input for f, replacing x everywhere it occurs with $\sqrt{x}$.

$f(g(x)) = 2\sqrt{x} + 3$

Algebraically, the composition of f with g is the function $y = 2\sqrt{x} + 3$.

In terms of transformations, $f \circ g$ represents the function $y = g(x)$ stretched vertically by a factor of 2 and translated 3 units up. Its domain is $\{x \in \mathbf{R} \mid x \geq 0\}$.

$$(g \circ f)(x) = g(\underbrace{f(x)}_{})$$

outer function $\quad$ inner function

When g is composed with f, take the output from the inner function f and use it as the input for the outer function g.

$$x \xrightarrow{f} f(x) \xrightarrow{g} g(f(x))$$
$$f(x) = 2x + 3$$
$$\begin{aligned} g(f(x)) &= g(2x + 3) \\ &= \sqrt{(2x + 3)} \end{aligned}$$

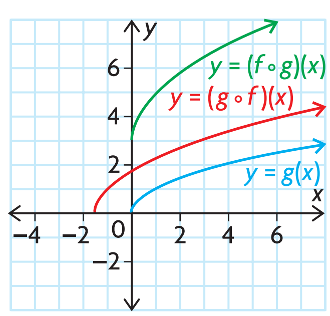
The output from f is the expression $2x + 3$. Use this as the input for g, and replace x everywhere it occurs with $2x + 3$.

$g(f(x)) = \sqrt{2x + 3}$

Algebraically, the composition of g with f is the function

$g(f(x)) = \sqrt{2x + 3} = \sqrt{2(x + 1.5)}$

In terms of transformations, $y = g(x)$ is compressed horizontally by a factor of $\frac{1}{2}$ and translated 1.5 units to the left. Its domain is $\left\{x \in \mathbf{R} \mid x \geq -\frac{3}{2}\right\}$.



Clearly, the expressions for $y = (f \circ g)(x)$ and $y = (g \circ f)(x)$ are different. Comparing their graphs illustrates the result of applying different sequences of transformations to $y = g(x)$.

$(f \circ g)(x) \neq (g \circ f)(x)$. The compositions of these two functions generate different answers depending on the order of the composition.

EXAMPLE 3 Reasoning about the domain of a composite function

Let $f(x) = \log_2 x$ and $g(x) = x + 4$.

a) Determine $f \circ g$, and find its domain.

b) What is the relationship between the domain of $f \circ g$ and the domain and range of f and g?

Solution

a) $(f \circ g)(x) = f(g(x))$

$= f(x + 4)$

$= \log_2(x + 4)$

Use the output for g as the input for f.

Since $x + 4 > 0 \Rightarrow x > -4$.

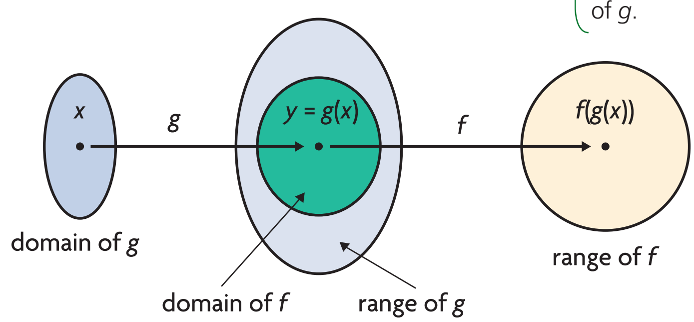
The domain of $f \circ g$ is $x \in (-4, \infty)$.

The domain of a logarithmic function with base a contains only positive real numbers, so the expression $x + 4$ must be greater than zero.

b) Domain of f: $x \in (0, \infty)$ Range of f: $y \in \mathbf{R}$

Domain of g: $x \in \mathbf{R}$ Range of g: $y \in \mathbf{R}$

Looking at the domain of $f \circ g$, we can see that it is not equal to either the domain of f or the domain of g.



Recall that the output values (range of g) for $y = g(x)$, are used as the input values (domain) for f.

In this example, the domain of f is $x > 0$ and the domain of g is $x \in \mathbf{R}$, so the only y-values of g that can be used occur when $g(x) > 0$.

Since $g(x) = x + 4$, $x + 4 > 0$

$x > -4$

The domain of $f \circ g$ is the set of values, x, in the domain of g for which $g(x)$ is in the domain of f.

EXAMPLE 4 Reasoning about a function composed with its inverse

Show that, if $f(x) = \frac{1}{x-2}$ then $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x)$.

Solution

$$\begin{aligned} x &= \frac{1}{y-2} \\ x(y-2) &= 1 \\ y - 2 &= \frac{1}{x} \\ y &= \frac{1}{x} + 2 \text{ or } f^{-1}(x) = \frac{1}{x} + 2 \end{aligned}$$

To find the inverse of f, switch x and y and then solve for y.

$$\begin{aligned} (f \circ f^{-1})(x) &= f(f^{-1}(x)) \\ &= f\left(\frac{1}{x} + 2\right) \\ &= \frac{1}{\left(\frac{1}{x} + 2\right) - 2} \\ &= \frac{1}{\left(\frac{1}{x}\right)} \\ &= x \end{aligned}$$

So, $(f \circ f^{-1})(x) = x$

The composition of f with its inverse maps a number in the domain of f onto itself. In other words, the result of this composition is the line $y = x$.

$$\begin{aligned} (f^{-1} \circ f)(x) &= f^{-1}(f(x)) \\ &= f^{-1}\left(\frac{1}{x-2}\right) \\ &= \frac{1}{\left(\frac{1}{x-2}\right)} + 2 \\ &= x - 2 + 2 \\ &= x \end{aligned}$$

The composition of f^{-1} with f maps a number in the domain of f^{-1} onto itself. In other words, the result of this composition is also the line $y = x$.

So, $(f^{-1} \circ f)(x) = x$

Therefore, $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x)$

Even though the order of the functions in the composition is reversed, the results are the same.

EXAMPLE 5 Working backward to decompose a composite function

Given $h(x) = |x^3 - 1|$, find two functions, f and g, such that $h = f \circ g$.

Solution

To evaluate h for any value of x, take that value, cube it, and subtract 1. This defines a sequence of operations for the inner function. Then, take the absolute value. This defines the outer function.

Let $g(x) = x^3 - 1$ and $f(x) = |x|$.

When evaluating the composition of f with g, you start by evaluating g for some value of x. So, it makes sense to define the inner function g that h performs on any input value. Then define the outer function f to represent the remaining operation(s) required by h.

$$\begin{aligned}
\text{Then } (f \circ g)(x) &= f(g(x)) \\
&= f(x^3 - 1) \\
&= |x^3 - 1| \\
&= h(x)
\end{aligned}$$

$$h(x) = (f \circ g)(x)$$

Another solution would be to let $g(x) = x^3$ and $f(x) = |x - 1|$.

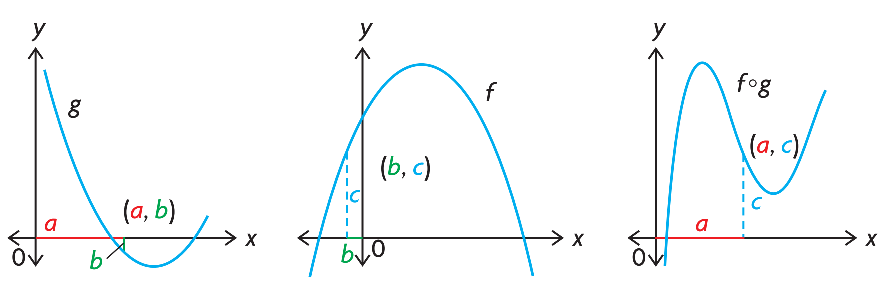
In Summary

Key Idea

- Two functions, f and g, can be combined using a process called composition, which can be represented by $f(g(x))$. The output for the inner function g is used as the input for the outer function f. The function $f(g(x))$ can be denoted by $(f \circ g)(x)$.

Need to Know

- Algebraically, the composition of f with g is denoted by $(f \circ g)(x)$, whereas the composition of g with f is denoted by $(g \circ f)(x)$. In most cases, $(f \circ g)(x) \neq (g \circ f)(x)$ because the order in which the functions are composed matters.
- Let $(a, b) \in g$ and $(b, c) \in f$. Then $(a, c) \in f \circ g$. A point in $f \circ g$ exists where an element in the range of g is also in the domain of f. The function $f \circ g$ exists only when the range of g overlaps the domain of f.

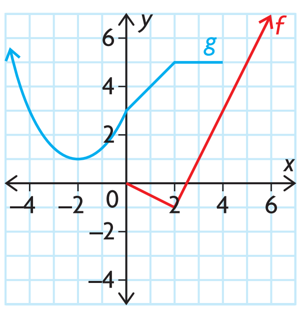


- The domain of $(f \circ g)(x)$ is a subset of the domain of g. It is the set of values, x, in the domain of g for which $g(x)$ is in the domain of f.
- If both f and f^{-1} are functions, then $(f^{-1} \circ f)(x) = x$ for all x in the domain of f, and $(f \circ f^{-1})(x) = x$ for all x in the domain of f^{-1}.

CHECK Your Understanding

1. Use $f(x) = 2x - 3$ and $g(x) = 1 - x^2$ to evaluate the following expressions.
   a) $f(g(0))$
   b) $g(f(4))$
   c) $(f \circ g)(-8)$
   d) $(g \circ g)\left(\frac{1}{2}\right)$
   e) $(f \circ f^{-1})(1)$
   f) $(g \circ g)(2)$

2. Given $f = \{(0, 1), (1, 2), (2, 5), (3, 10)\}$ and $g = \{(2, 0), (3, 1), (4, 2), (5, 3), (6, 4)\}$, determine the following values.
   a) $(g \circ f)(2)$
   b) $(f \circ f)(1)$
   c) $(f \circ g)(5)$
   d) $(f \circ g)(0)$
   e) $(f \circ f^{-1})(2)$
   f) $(g^{-1} \circ f)(1)$



3. Use the graphs of f and g to evaluate each expression.
   a) $f(g(2))$
   b) $g(f(4))$
   c) $(g \circ g)(-2)$
   d) $(f \circ f)(2)$

4. For a car travelling at a constant speed of 80 km/h, the distance driven, d kilometres, is represented by $d(t) = 80t$, where t is the time in hours. The cost of gasoline, in dollars, for the drive is represented by $C(d) = 0.09d$.
   a) Determine $C(d(5))$ numerically, and interpret your result.
   b) Describe the relationship represented by $C(d(t))$.

PRACTISING

5. **K** In each case, functions f and g are defined for $x \in \mathbf{R}$. For each pair of functions, determine the expression and the domain of $f(g(x))$ and $g(f(x))$. Graph each result.
   a) $f(x) = 3x^2, g(x) = x - 1$
   b) $f(x) = 2x^2 + x, g(x) = x^2 + 1$
   c) $f(x) = 2x^3 - 3x^2 + x - 1, g(x) = 2x - 1$
   d) $f(x) = x^4 - x^2, g(x) = x + 1$
   e) $f(x) = \sin x, g(x) = 4x$
   f) $f(x) = |x| - 2, g(x) = x + 5$

6. For each of the following,
   - determine the defining equation for $f \circ g$ and $g \circ f$
   - determine the domain and range of $f \circ g$ and $g \circ f$

   a) $f(x) = 3x, g(x) = \sqrt{x - 4}$
   b) $f(x) = \sqrt{x}, g(x) = 3x + 1$
   c) $f(x) = \sqrt{4 - x^2}, g(x) = x^2$
   d) $f(x) = 2^x, g(x) = \sqrt{x - 1}$
   e) $f(x) = 10^x, g(x) = \log x$
   f) $f(x) = \sin x, g(x) = 5^{2x} + 1$

7. For each function h, find two functions, f and g, such that $h(x) = f(g(x))$.

a) $h(x) = \sqrt{x^2 + 6}$

b) $h(x) = (5x - 8)^6$

c) $h(x) = 2^{(6x+7)}$

d) $h(x) = \dfrac{1}{x^3 - 7x + 2}$

e) $h(x) = \sin^2(10x + 5)$

f) $h(x) = \sqrt[3]{(x + 4)^2}$

8. **a)** Let $f(x) = 2x - 1$ and $g(x) = x^2$. Determine $(f \circ g)(x)$.

b) Graph f, g, and $f \circ g$ on the same set of axes.

c) Describe the graph of $f \circ g$ as a transformation of the graph of $y = g(x)$.

9. Let $f(x) = 2x - 1$ and $g(x) = 3x + 2$.

a) Determine $f(g(x))$, and describe its graph as a transformation of $g(x)$.

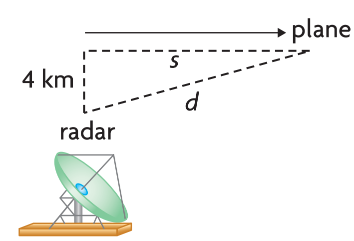
b) Determine $g(f(x))$, and describe its graph as a transformation of $f(x)$.

10. A

A banquet hall charges \$975 to rent a reception room, plus \$39.95 per person. Next month, however, the banquet hall will be offering a 20% discount off the total bill. Express this discounted cost as a function of the number of people attending.

11. The function $f(x) = 0.08x$ represents the sales tax owed on a purchase with a selling price of x dollars, and the function $g(x) = 0.75x$ represents the sale price of an item with a price tag of x dollars during a 25% off sale. Write a function that represents the sales tax owed on an item with a price tag of x dollars during a 25% off sale.

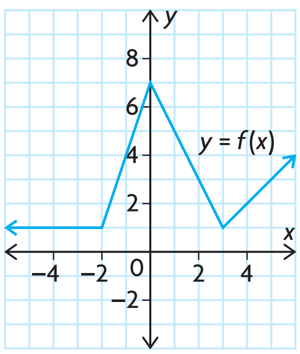
12. An airplane passes directly over a radar station at time $t = 0$. The plane maintains an altitude of 4 km and is flying at a speed of 560 km/h. Let d represent the distance from the radar station to the plane, and let s represent the horizontal distance travelled by the plane since it passed over the radar station.



a) Express d as a function of s, and s as a function of t.

b) Use composition to express the distance between the plane and the radar station as a function of time.

13. In a vehicle test lab, the speed of a car, v kilometres per hour, at a time of t hours is represented by $v(t) = 40 + 3t + t^2$. The rate of gasoline consumption of the car, c litres per kilometre, at a speed of v kilometres per hour is represented by $c(v) = \left(\frac{v}{500} - 0.1\right)^2 + 0.15$. Determine algebraically $c(v(t))$, the rate of gasoline consumption as a function of time. Determine, using technology, the time when the car is running most economically during a 4 h simulation.



14. Given the graph of $y = f(x)$ shown and the functions below, match **T** the correct composition with each graph. Justify your choices.

i) $g(x) = x + 3$ **iii)** $h(x) = x - 3$ **v)** $k(x) = -x$

ii) $m(x) = 2x$ **iv)** $n(x) = -0.5x$ **vi)** $p(x) = x - 4$

a) $y = (f \circ g)(x)$ **g)** $y = (g \circ f)(x)$

b) $y = (f \circ h)(x)$ **h)** $y = (h \circ f)(x)$

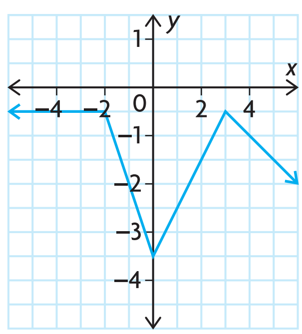
c) $y = (f \circ k)(x)$ **i)** $y = (k \circ f)(x)$

d) $y = (f \circ m)(x)$ **j)** $y = (m \circ f)(x)$

e) $y = (f \circ n)(x)$ **k)** $y = (n \circ f)(x)$

f) $y = (f \circ p)(x)$ **l)** $y = (p \circ f)(x)$

A



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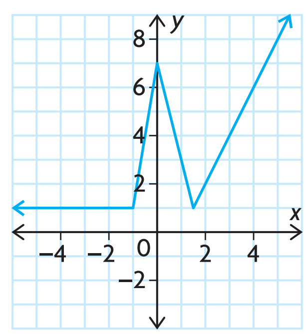

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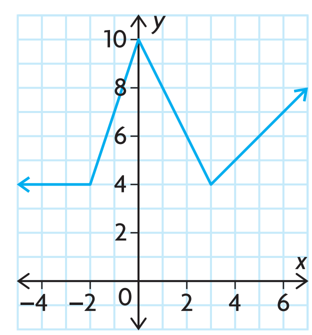

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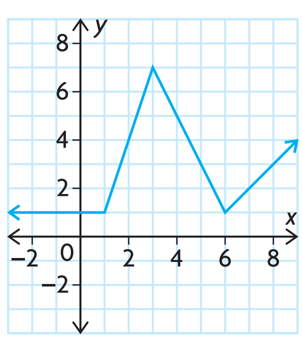

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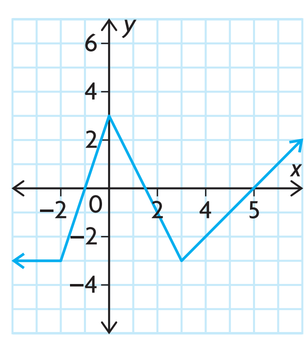

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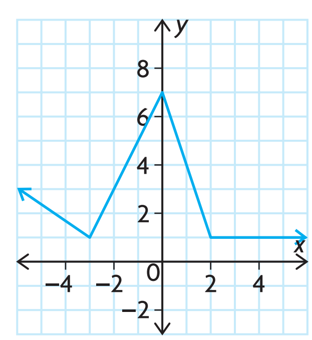

Sum	Product
$y = \frac{4}{x-3} + 1$	
Quotient	Composition

15. Find two functions, f and g, to express the given function in the centre **C** box of the chart in each way shown.

Extending

16. a) If $y = 3x - 2$, $x = 3t + 2$, and $t = 3k - 2$, find an expression for $y = f(k)$.

b) Express y as a function of k if $y = 2x + 5$, $x = \sqrt{3t - 1}$, and $t = 3k - 5$.