9.6 Techniques for Solving Equations and Inequalities

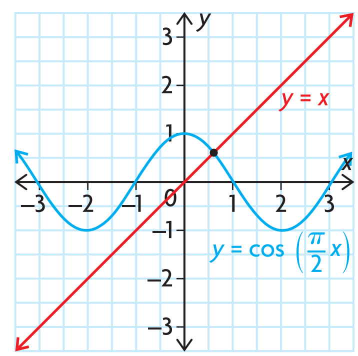
GOAL

Solve equations and inequalities that involve combinations of functions using a variety of techniques.

YOU WILL NEED

- graphing calculator

LEARN ABOUT the Math



On the graph are the functions $y = \cos\left(\frac{\pi}{2}x\right)$ and $y = x$. The point of intersection of the two functions is the point where $\cos\left(\frac{\pi}{2}x\right) = x$.

? How can the equation $\cos\left(\frac{\pi}{2}x\right) = x$ be solved to determine the point of intersection of these two functions?

EXAMPLE 1 Selecting tools and strategies to solve an equation

Solve the equation $\cos\left(\frac{\pi}{2}x\right) = x$ to the nearest hundredth.

Solution A: Selecting a guess and improvement strategy that involves a numerical approach

$$\cos\left(\frac{\pi}{2}x\right) = x$$

Using the given graph, the point of intersection looks like it occurs when x is about 0.5.

$$\cos\left(\frac{\pi}{2}x\right) - x = 0$$

Subtract x from both sides of the equation so that one side is equal to zero.

$$\cos\left(\frac{\pi}{2}(0.5)\right) - 0.5$$
$$= \cos\left(\frac{\pi}{4}\right) - 0.5$$
$$= \frac{1}{\sqrt{2}} - 0.5$$
$$\doteq 0.207$$

Check the estimate by substituting the value $x = 0.5$ into the equation.

0.207 is close to zero, but there may be some other values close to 0.5 that give a better answer.

When $x = 0.4$,

$$\cos\left(\frac{\pi}{2}(0.4)\right) - 0.4 \doteq 0.409$$

Repeat the process for $x = 0.4$.

The result is farther away from zero than the previous estimate, so try a number larger than 0.5.

When $x = 0.6$,

$$\cos\left(\frac{\pi}{2}(0.6)\right) - 0.6 \doteq -0.0122$$

Repeat the process for $x = 0.6$.

The result is closer to zero than the previous two estimates, but is a little below zero. Try a number a bit smaller than 0.6.

When $x = 0.59$,

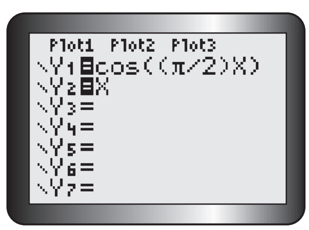
$$\cos\left(\frac{\pi}{2}(0.59)\right) - 0.59 \doteq 0.0104$$

$$\cos\left(\frac{\pi}{2}x\right) = x \text{ when } x \doteq 0.59$$

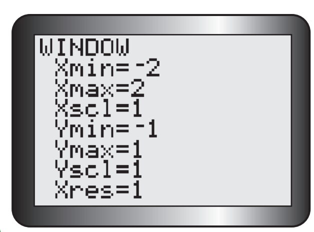
Repeat the process for $x = 0.59$.

$x = 0.59$ is a much better answer because it gives a y-value that is almost equal to zero.

Solution B: Selecting a graphical strategy that involves the points of intersection

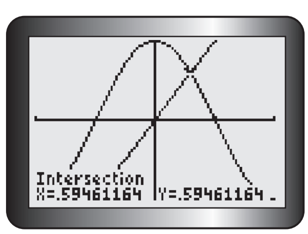


Enter the function equations $y = \cos\left(\frac{\pi}{2}x\right)$ and $y = x$ into the equation editor on a graphing calculator, as Y1 and Y2. Graph using a suitable window in radian mode.



Tech Support

For help using a graphing calculator to determine points of intersection, see Technical Appendix, T-12.



Use the intersect operation to determine the point of intersection.

$$\cos\left(\frac{\pi}{2}x\right) = x \text{ when } x \doteq 0.59$$

This is the only point of intersection since $y = \cos\left(\frac{\pi}{2}x\right)$ alternates between 1 and -1, while $y = x$ has the following end behaviours:

As $x \to \infty$, $y \to \infty$, and as $x \to -\infty$, $y \to -\infty$.

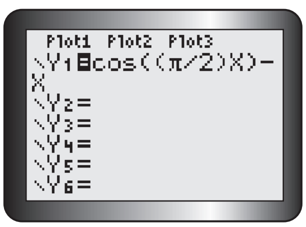
Solution C: Selecting a graphical strategy that involves the zeros

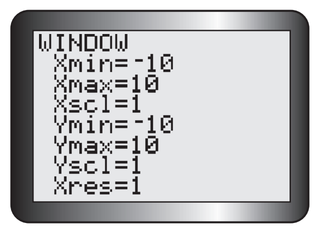
Recall that solving for the roots of an equation is related to finding the zeros of a corresponding function.

$\cos\left(\frac{\pi}{2}x\right) = x$ is equivalent to

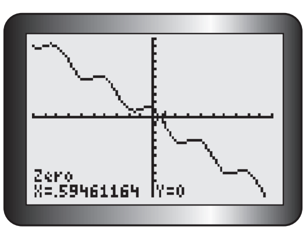
$\cos\left(\frac{\pi}{2}x\right) - x = 0$

Sketch the graph of $y = \cos\left(\frac{\pi}{2}x\right) - x$ on a graphing calculator, using a suitable window.





Use the zero operation to determine the function's zeros.



$\cos\left(\frac{\pi}{2}x\right) = x$ when $x \doteq 0.59$

Tech Support

For help using a graphing calculator to determine the zeros of a function, see Technical Appendix, T-8.

Reflecting

A. What are the advantages of using a guess and improvement strategy versus a graphing strategy? What are the disadvantages?

B. When using a guess and improvement strategy, how will you know when a given value of x gives you an accurate answer?

C. Which graphical strategy do you prefer? Explain.

APPLY the Math

EXAMPLE 2 Using an equation to solve a problem

According to data collected from 1996 to 2001, the average price of a new condominium in Toronto was \$144 144 in 2001 and increased by 6.6% each year. A new condominium in Regina cost \$72 500 on average, but prices were growing by 10% per year there. If these trends continue, when will a new condominium in Regina be the same price as one in Toronto?

Solution

Let x be the number of years since 2001.
Let y be the price of a new condominium.

Toronto: $y = 144\,144(1.066)^x$

Regina: $y = 72\,500(1.10)^x$

These are the exponential functions that model the average price of a new condominium in Toronto and Regina since 2001.

Solve $144\,144(1.066)^x = 72\,500(1.10)^x$.

To determine when the prices are the same, set the two functions equal to each other.

$$\frac{144\,144(1.066)^x}{72\,500} = \frac{72\,500(1.10)^x}{72\,500}$$

$$1.9882(1.066)^x \doteq (1.10)^x$$

$$\frac{1.9882\cancel{(1.066)^x}}{\cancel{(1.066)^x}} = \frac{(1.10)^x}{(1.066)^x}$$

$$1.9882 = \left(\frac{1.10}{1.066}\right)^x$$

This exponential equation can be solved algebraically.

Divide both sides by 72 500.

Divide both sides by 1.066^x.

$$\log(1.9882) = \log\left(\frac{1.10}{1.066}\right)^x$$

$$\log(1.9882) = x\log\left(\frac{1.10}{1.066}\right)$$

$$\log(1.9882) = x(\log(1.10) - \log(1.066))$$

$$\frac{\log(1.9882)}{\log(1.10) - \log(1.066)} = \frac{x\cancel{\log(1.10) - \log(1.066))}}{\cancel{(\log(1.10) - \log(1.066))}}$$

$$21.89 \doteq x$$

Take the log of both sides.

Rewrite the right side using the logarithm laws.

Divide both sides by $\log(1.10) - \log(1.066)$.

Evaluate the left side.

If these trends continue, the price of a new condominium in Regina will be the same as the price of a new condominium in Toronto by the end of the year 2023.

$2001 + 21.89 \doteq 2023$

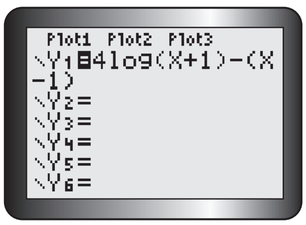
EXAMPLE 3 Selecting a graphing strategy to solve an inequality

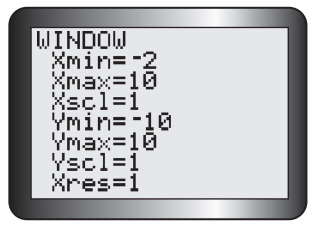
Given $f(x) = 4 \log (x + 1)$ and $g(x) = x - 1$, determine all values of x such that $f(x) > g(x)$.

Solution A: Using a single function and comparing its position to the *x*-axis

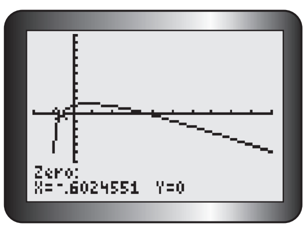
If $f(x) > g(x)$, then $f(x) - g(x) > 0$.

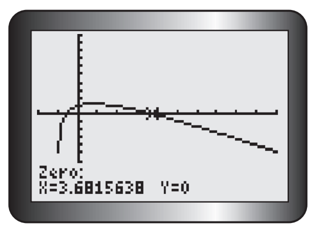
Let $y_1 = (f - g)(x) = 4 \log (x + 1) - (x - 1)$.





Enter the function into the equation editor, and graph the function using an appropriate window. In this example, $x > -1$ since the log function is undefined for negative values.



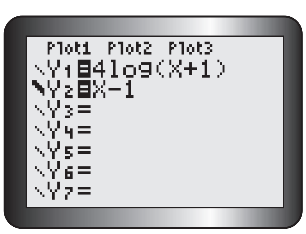


Use the zero operation to determine its zeros.

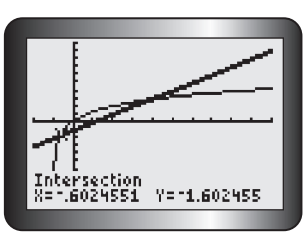
$f(x) > g(x)$ when $x \in (-0.602, 3.681)$.

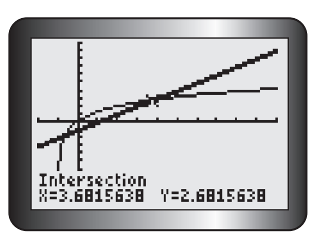
$f(x) - g(x) > 0$ when its graph is above the x-axis. This occurs in the interval between the two zeros.

Solution B: Using both functions and comparing the position of one to the other



Enter the two functions, f and g, into Y1 and Y2, respectively, in the equation editor on a graphing calculator. Use a bold line for Y2.





Determine the points of intersection using the intersect operation.

$f(x) > g(x)$ when $x \in (-0.602, 3.681)$.

This means that f lies above g in the interval between the two intersection points.

In Summary

Key Ideas

- The equation $f(x) = g(x)$ can be solved using a guess and improvement strategy. Estimate where the intersection of $f(x)$ and $g(x)$ will occur, and substitute this value into both sides of the equation. Based on the outcome, adjust your estimate. Repeat this process until the desired degree of accuracy is found.
- If graphing technology is available, the equation $f(x) = g(x)$ can be solved by graphing the two functions and using the intersect operation to determine the point of intersection.
- The equation $f(x) = g(x)$ can also be solved by rewriting the equation in the form $f(x) - g(x) = 0$ to obtain the corresponding function, $h(x) = f(x) - g(x)$. The zeros of this function are also the roots of the equation. These can be determined using a guess and improvement strategy when graphing technology is not available. Graphing technology can also be used to graph the function $h(x) = f(x) - g(x)$ and determine its zeros using the zero operation.
- Inequalities can be solved by using these strategies to solve the corresponding equation, and then selecting the intervals that satisfy the inequality.

Need to Know

- The method used to solve equations and inequalities depends on the degree of accuracy required and the access to graphing technology. A solution using graphing technology will usually result in a closer approximation to the root (zero) of the equation than a solution generated by a numerical strategy with the aid of a scientific calculator.
- The difference between the solution to a strict inequality, $f(x) > g(x)$, and an inclusive inequality, $f(x) \geq g(x)$, is that the value of each root (zero) is included in the solution to the inclusive inequality.

CHECK Your Understanding

1. For each graph shown below, state the solution to each of the following:

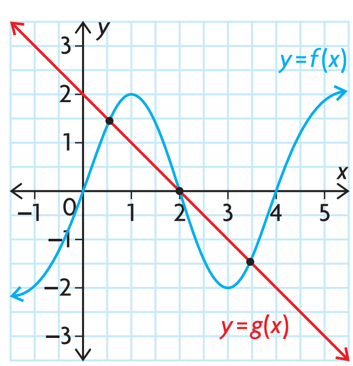
a) $f(x) = g(x)$

b) $f(x) > g(x)$

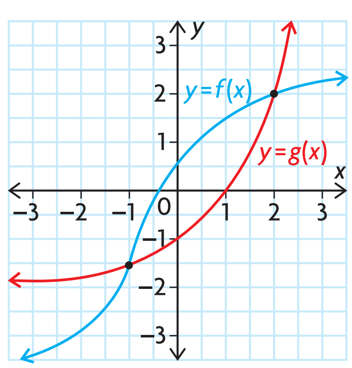
c) $f(x) \leq g(x)$

d) $f(x) \geq g(x)$

i)



ii)

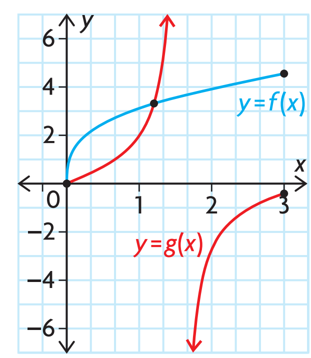


2. Use a guess and improvement strategy to determine the best one-decimal-place approximation to the solution of each equation in the interval provided.
   a) $3 = 2^{2x}$, when $x \in [0, 2]$
   b) $0 = \sin(0.25x^2)$, when $x \in [0, 5]$
   c) $3x = 0.5x^3$, when $x \in [-8, -1]$
   d) $\cos x = x$, when $x \in \left[0, \frac{\pi}{2}\right]$

3. Use graphing technology to determine the solution to $f(x) = g(x)$, where $f(x) = 2\sqrt{x + 3}$ and $g(x) = x^2 + 1$, in two different ways.

PRACTISING

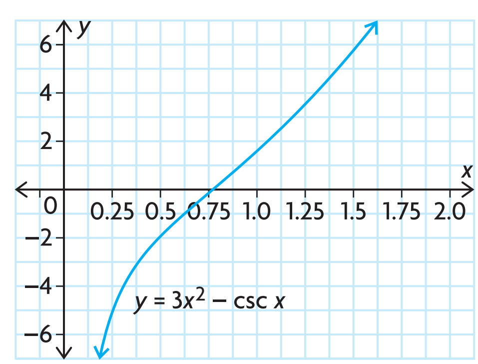
4. In the graph shown, $f(x) = 3\sqrt[3]{x}$ and $g(x) = \tan x$. State the values
K of x in the interval [0, 3] for which $f(x) < g(x)$, $f(x) = g(x)$, and $f(x) > g(x)$. Express the values to the nearest tenth.



5. Solve each of the following equations for x in the given interval, using a guess and improvement strategy. Express your answers to the nearest tenth.
   a) $5 \sec x = -x^2, 0 \le x \le \pi$
   b) $\sin^3 x = \sqrt{x} - 1, 0 \le x \le \pi$
   c) $5^x = x^5, -2 \le x \le 2$
   d) $\cos x = \frac{1}{x}, -4 \le x \le 0$
   e) $\log(x) = (x - 10)^2 + 1, 0 \le x \le 10$
   f) $\sin(2\pi x) = -4x^2 + 16x - 12, 0 \le x \le 5$

6. Use graphing technology to solve each of the following equations. Round to two decimal places, if necessary.
   a) $2^x - 1 = \log(x + 2)$
   b) $\sqrt{x + 5} = x^2$
   c) $\sqrt{x + 3} - 5 = -x^4$
   d) $\sqrt[3]{\sin x} = 2x^3$ for x in the interval $-3 \le x \le 3$
   e) $\cos(2\pi x) = -x + 0.5$ in the interval $0 \le x \le 1$
   f) $\tan(2\pi x) = 2\sin(3\pi x)$ in the interval $0 \le x \le 1$

7. To solve the equation $-\csc x = -3x^2$ for x in the interval $0 \le x \le 2$, the graph shown can be used. Determine the coordinates of the point where the graphs of the functions $f(x) = -\csc x$ and $g(x) = -3x^2$ intersect in the interval $0 \le x \le 2$.



8. **A** Two jurisdictions in Canada and the United States are attempting to decrease the numbers of mountain pine beetles that have been damaging their national forests. A section of forest under study in British Columbia at the beginning of 1997 had an estimated 2.3 million of the pests, while there were about 1.95 million of the pests in a similar-sized section of forest in the state of Washington. British Columbia has been decreasing the number of mountain pine beetles by 4% per year, while Washington has been decreasing the number by 3% per year. When will there be about the same number of pests in the sections of forest under study in each jurisdiction?

9. Solve each of the following inequalities using graphing technology. State your solutions using interval notation, rounding to the nearest hundredth as required.

a) $2x^2 < 2^x$

b) $\log (x + 1) \geq x^3$

c) $\left(\frac{1}{2}\right)^x > \frac{1}{x}$

d) $\sin (\pi x) > \cos (2\pi x)$, where $x \in [0, 1]$

e) $\cos (\pi x) \leq \left(\frac{1}{10}\right)^x$, where $x \in [0, 2]$

f) $\tan (\pi x) > \sqrt{x}$, where $x \in [0, 1]$

10. **T** Give an example of two functions, f and g, such that $f(x) > g(x)$ when $x \in [-4, -2]$ or $x \in [1, \infty)$.

11. Give an example of two functions, f and g, such that $f(x) > 0$ when $x \in [-5, 5]$ and $f(x) > g(x)$ when $x \in [-4, 5]$.

12. Two of the solutions to the equation $a \cos x = bx^3 + 6$, where a and b are integers, are $x = -1.2$ and $x = -0.7$. These solutions are rounded to the nearest tenth. What are the values of a and b?

13. **C** Construct a flow chart to describe the process of finding the solutions to an equation using your preferred strategy.

Extending

14. Determine the general solution to the equation $\tan (0.5\pi x) = 2 \sin (\pi x)$.

15. Determine the general solution to the inequality $\sin (\pi x) > 0$.