# 9.7 Modelling with Functions

**GOAL**

Use a variety of functions to model real-life situations.

**YOU WILL NEED**

- graphing calculator or graphing software/dynamic statistical software

## *LEARN ABOUT the Math*

About 5000 people live in Sanjay's town. One person in his school came back from their March Break trip to Florida with a virus. A week later, 70 additional people have the virus, and doctors in the town estimate that about 8% of the town's residents will eventually get this virus.

**?** What types of functions could be used to model the spread of the virus in this town?

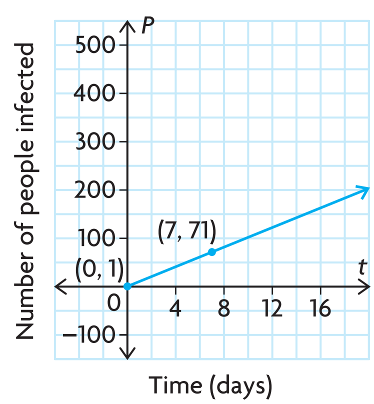
### EXAMPLE 1 Selecting a function to model the situation

Select an appropriate function to model the spread of the virus in Sanjay's town.

#### Solution A: Selecting a linear model

Use the given data to sketch a graph.

| Time, $t$ (days) | People Infected, $P$ |
|---|---|
| 0 | 1 |
| 7 | 71 |




The general equation of the linear model is $y = mx + b$, where $m$ is the slope of the line and $b$ is the $y$-intercept.

Time, $t$, is the independent variable. The number of people infected, $P$, is the dependent variable.

In this case, the vertical or $y$-intercept is 1 and the slope is

$$\frac{\Delta P}{\Delta t} = \frac{71 - 1}{7 - 0}$$
$$= 10$$

Two points are sufficient to determine the equation of a line.

The linear model is $P(t) = 10t + 1$ and predicts that the number of people infected by the virus will grow at a constant rate of 10 people per day.

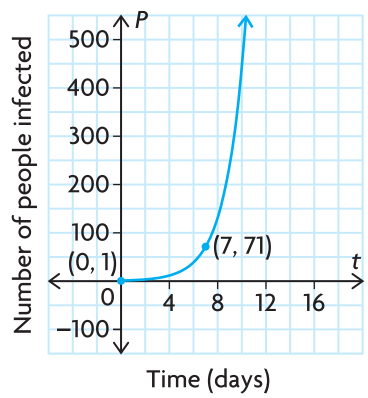
$$P(t) = 400$$
$$400 = 10t + 1$$
$$399 = 10t$$
$$39.9 = t$$

8% of 5000 is 400.
At a rate of 10 people per day, it will take about 40 days for the virus to spread to the expected number of 400 people.

## Solution B: Selecting an exponential model

Use the given data to sketch a graph.

| Time, $t$ (days) | People Infected, $P$ |
|---|---|
| 0 | 1 |
| 7 | 71 |




The general equation of the exponential model is $y = ab^t$, where $a$ is the initial value, or $y$-intercept, and $b$ is $(1 + \text{growth rate})$.

Time, $t$, is the independent variable. The number of people infected, $P$, is the dependent variable.

$$P(t) = P_0(b)^t$$

Substituting gives

$$71 = 1(b)^7$$
$$71 = b^7$$
$$\sqrt[7]{71} = b$$
$$1.8385 \doteq b$$

Two points are enough to determine an exponential model. The initial value of the exponential function is $P_0 = 1$, and we know that $P(7) = 71$.

The exponential model is $P(t) = 1(1.8385)^t$.

The exponential model predicts slow initial growth followed by much faster growth.

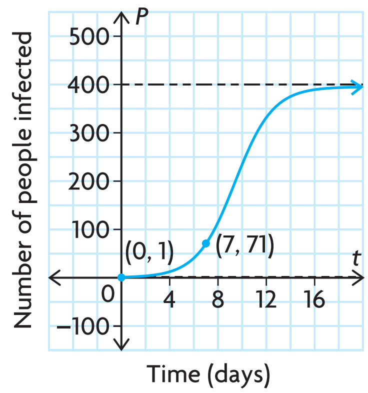
$$P(t) = 400$$
$$400 = 1(1.8385)^t.$$
$$\log(400) = \log(1.8385)^t$$
$$\frac{\log(400)}{\log(1.8385)} = \frac{t\,\cancel{\log(1.8385)}^{1}}{\cancel{\log(1.8385)}_{1}}$$
$$9.84 \doteq t$$

This model predicts that it will take about 10 days for the virus to infect the expected number of 400 people.

### Solution C: Selecting a logistic model

Use the given data to sketch a graph.

| Time, $t$ (days) | People Infected, $P$ |
|---|---|
| 0 | 1 |
| 7 | 71 |




The general equation of the logistic model is $P(t) = \dfrac{c}{1 + ab^t}$ where $c$ is the carrying capacity, or maximum value, that the function attains.

Time, $t$, is the independent variable. The number of people infected, $P$, is the dependent variable.

The carrying capacity, $c$, or maximum number of people infected, is 8% of $5000 = 400$.

Substituting $P(0) = 1$ gives

$$1 = \frac{400}{1 + ab^0}$$

$$1 = \frac{400}{1 + a}$$

$$a = 399$$

Substituting $P(7) = 71$ gives

$$71 = \frac{400}{1 + 399b^7}$$

$$1 + 399b^7 = \frac{400}{71}$$

$$399b^7 \doteq 5.6338 - 1$$

$$b^7 \doteq 0.011\,614$$

$$b \doteq 0.5291$$

The parameters $a$ and $b$ can be determined if two points on the function are known.

The logistic model is $P(t) = \dfrac{400}{1 + 399(0.5291)^t}$.

The logistic model predicts slow growth followed by rapid growth, and then a slowing of the growth rate again as the maximum number of infected people nears 400.

The graph approaches a horizontal asymptote at $P = 400$ when $t$ is close to 12.

This model predicts that it will take about 12 days for the virus to infect the expected number of 400 people.

## Reflecting

**A.** Compare the growth curves for the three mathematical models. How do the graphs differ? How are they similar?

**B.** How do the growth rates for the three mathematical models compare?

**C.** No mathematical model is perfect; what we hope for is a useful description of the situation. Which of these models do you think is the least realistic, and which one the most realistic? Why?

**D.** What could you do in a situation like this to improve the accuracy of your mathematical model?

**E.** Are there any other types of functions that you think could be used to model this situation? Explain.

## *APPLY the Math*

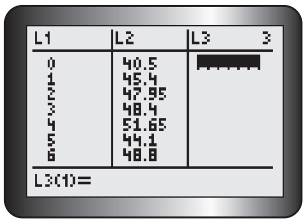
### EXAMPLE 2 Selecting a function model to fit to a data set

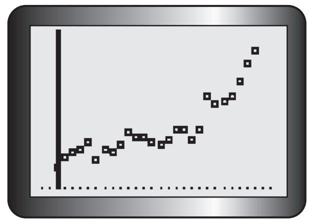
The table shows the median annual price for unleaded gasoline in Toronto for a 26-year period. Determine a mathematical model for the data, compare the values with the given values, and use the values to predict the median price of unleaded gasoline in 2010.

| Year | Years since 1981 | Price (cents/L) |
|---|---|---|
| 1981 | 0 | 40.5 |
| 1982 | 1 | 45.4 |
| 1983 | 2 | 47.95 |
| 1984 | 3 | 48.4 |
| 1985 | 4 | 51.65 |
| 1986 | 5 | 44.1 |
| 1987 | 6 | 48.8 |
| 1988 | 7 | 47.6 |
| 1989 | 8 | 51.5 |
| 1990 | 9 | 56.55 |
| 1991 | 10 | 54.4 |
| 1992 | 11 | 54.35 |
| 1993 | 12 | 52.3 |

| Year | Years since 1981 | Price (cents/L) |
|---|---|---|
| 1994 | 13 | 50.65 |
| 1995 | 14 | 53.5 |
| 1996 | 15 | 58.0 |
| 1997 | 16 | 58.05 |
| 1998 | 17 | 53.45 |
| 1999 | 18 | 58.1 |
| 2000 | 19 | 72.75 |
| 2001 | 20 | 69.85 |
| 2002 | 21 | 70.85 |
| 2003 | 22 | 72.45 |
| 2004 | 23 | 79.55 |
| 2005 | 24 | 88.25 |
| 2006 | 25 | 93.65 |

## Solution A: Selecting a cubic model using regression on a graphing calculator



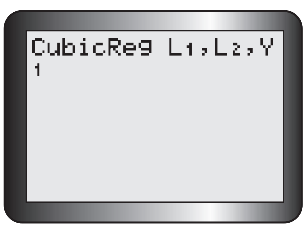



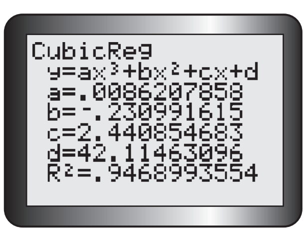
Enter the data into lists, and create a scatter plot.

The scatter plot clearly shows a non-linear trend. The graph increases, so possible functions include an exponential model, a quadratic model, and a cubic model.

Other functions are possible too, but a relatively simple model is preferred for ease of computation and use.

Since the data indicate that gas prices rose, then dropped a little, and then rose again, try a cubic model.



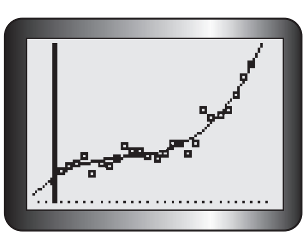




Perform a cubic regression on L1 and L2.

Note that the value of $R^2$ in the calculator output is 0.947. This means that 94.7% of the variation in gasoline prices is explained by our mathematical model.

The output is displayed, and the coefficients in the cubic polynomial are rounded.

$$f(x) = 0.0086x^3 - 0.2310x^2 + 2.4409x + 42.1146$$



The regression curve fits the scatter plot well.

$$\begin{aligned} f(29) &= 0.0086(29)^3 - 0.2310(29)^2 \\ &\quad + 2.4409(29) + 42.1146 \\ &\doteq 128.38 \end{aligned}$$

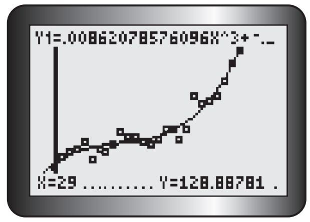
The year 2010 is 29 years after 1981, so substitute $t = 29$ to obtain a prediction of the price of gasoline.

**Tech Support**

For help creating a scatter plot using a graphing calculator, see Technical Appendix, T-11.

**Tech Support**

For help with regression to determine the equation of a curve of best fit using a graphing calculator, see Technical Appendix, T-11.

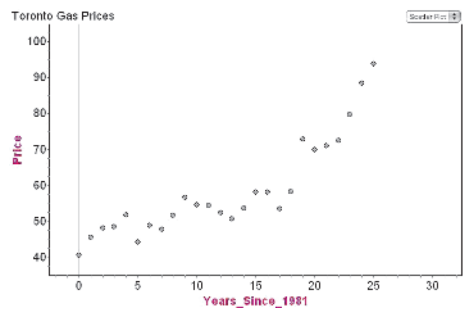



The cubic model predicts that the price for 2010 will be about \$1.28/L.

## Solution B: Selecting an exponential model using Fathom

**Tech Support**

For help creating a scatter plot using Fathom, see Technical Appendix, T-21.




Create a case table, and enter the years since 1981 as the independent variable and the price as the dependent variable.

Create a scatter plot of the data.

The exponential function model is of the form $P(t) = k + a(b)^t$, where $t$ is years since 1981 and $P$ is the price in cents.

Estimate the values of the parameters based on the scatter plot created and the data given.

$k$ is approximately 40.

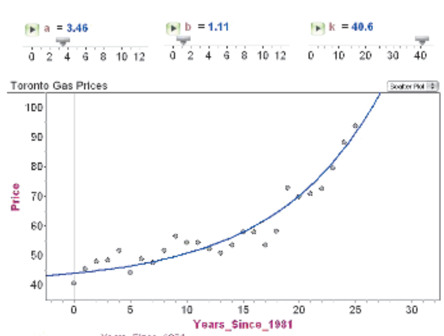
The shape of the scatter plot suggests that the horizontal asymptote for the exponential model is at about 40 cents per litre, so the parameter $k$ is approximately 40.

$b > 1$

$b$ must be greater than 1, since the exponential function increases.

**Tech Support**

For help graphing functions and creating sliders using Fathom, see Technical Appendix, T-21.




Create sliders for the parameters of $a$, $b$, and $k$, and enter the function as Price $= k + a(b)^t$. Adjust the sliders using a trial-and-error process until the curve fits the scatter plot.

An exponential model is

$$P(t) = 40.59 + 3.46(1.1134)^t.$$

$$P(29) = 40.59 + 3.46(1.1134)^{29}$$

$$= 118.57 \text{ cents per litre}$$

$$= \$1.19/\text{L}$$

The year 2010 is 29 years after 1981, so substitute $t = 29$ to obtain a prediction of the price of gasoline in 2010.

## In Summary

### Key Ideas

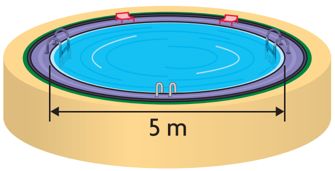
- A mathematical model is just that—a model. It will not be a perfect description of a real-life situation; but if it is a good model, then you will be able to use it to describe the real-life situation and make predictions.
- Increasing the amount of data you have for creating a mathematical model improves the accuracy of the model.
- A scatter plot gives you a visual representation of the data. Examining the scatter plot may give you an idea of what kind of function could be used to model the data. Graphing your mathematical model on the scatter plot is a visual way to confirm that it is a good fit.

### Need to Know

- If you have to choose between a simple function and a complicated function, and if both fit the data equally well, the simple function is generally preferred.
- The function you choose should make sense in the context of the problem; for the growth of a population, you may want to consider an exponential model or a logistic model.
- One way to compare mathematical models created using regression analysis is to examine the value of $R^2$. This is the fraction of the variation in the response variable ($y$), which is explained by the mathematical model based on the predictor variable ($x$).
- Mathematical models are useful for **interpolating**. They are not necessarily useful for **extrapolating** because they assume that the trend in the data will continue. Many factors can affect the relationship between the independent variable and the dependent variable and change the trend.
- It is often necessary to restrict the domain of a mathematical model to represent a realistic situation.

## *CHECK Your Understanding*

**1.** An above-ground swimming pool in the shape of a cylinder, with diameter 5 m, is filled at a constant rate to a depth of 1 m. It takes 4 h to fill the pool with a hose.




**a)** Make a graph showing volume of water in the pool as a function of time.

**b)** Determine the equation of a mathematical model for volume as a function of time.

**c)** When will the volume of the water be 8 $m^3$?

2. After being filled, the swimming pool in question 1 is accidentally punctured at the bottom and water leaks out. The volume of the pool reaches zero in 8 h. The volume of water remaining at time $t$ follows a quadratic model, with the minimum point (vertex) at the time when the last of the water drains out.

   a) Make a graph showing the volume of water in the pool versus time.
   b) Find the equation for the quadratic model.
   c) Use the model to predict the volume of water at the 2 h mark.
   d) What is the average rate of change in the volume of the water during the first 2 h?
   e) How does the rate of change in volume vary as time elapses?

3. An abandoned space station in orbit contains 200 $m^3$ of oxygen. It is punctured by a piece of space debris, and oxygen begins to leak out. After 4 h, there is 80 $m^3$ of oxygen remaining in the space station.

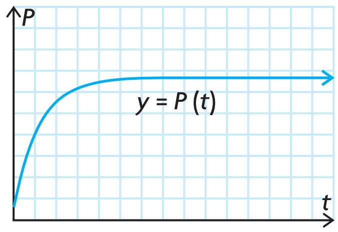
   a) Make a graph showing the two data points provided. Sketch two or three possible graphs that might show how volume decreases with time.
   b) The simplest model would be linear. Determine the equation of the linear model, and use this model to find the amount of time it will take for the last of the oxygen to escape.
   c) A more realistic model would be an exponential model, since the rate of change in volume is likely to be proportional to the volume of oxygen remaining. Determine the equation of an exponential model of the form $V(t) = a(b)^t$. Use this model to estimate the time it will take for 90% of the original volume of oxygen to escape.

## PRACTISING

4. **K** A lake in Northern Ontario has recovered from an acid spill that killed all of its trout. A restocking program puts 800 trout in the lake. Ten years later, the population is estimated to be 6000. The carrying capacity of the lake is believed to be 8000.

   a) Make a graph to show the given information. Extend the time scale to 20 years.
   b) Determine the parameters for a logistic model of the form $P(t) = \frac{c}{1 + a(b)^t}$ to model the growth of the trout population, and graph the function for $t \in [0, 20]$.
   c) Use the model to estimate the number of trout that were in the lake four years after restocking.
   d) Use the model to estimate the average rate of change in the number of trout over the first four years of the restocking program.

**5.** Consider again the population of trout in question 4. Another possible model for the trout situation is a transformed exponential function of the form $P(t) = c - a(b)^t$. A graph of this type of model, $y = P(t)$, is shown below.




**a)** What feature of the graph does the parameter $c$ represent? What is the value of $c$ for the trout population?
**b)** Determine the values of $a$ and $b$ by substituting the two known ordered pairs.
**c)** Graph this exponential model of the trout population for $t \in [0, 20]$.
**d)** Use the model to estimate the number of trout that were in the lake four years after restocking.
**e)** Use the model to estimate the average rate of change in the number of trout over the first four years of the restocking program.
**f)** Explain how this model differs from the logistic model in question 4.

**6.** Recall the cubic and exponential model equations for gasoline prices in Example 2. Which model more accurately calculates the current price of gasoline?

**7.** The following table shows the velocity of air, in litres per second, of a typical person's breathing while at rest.

| **Time (s)** | 0 | 0.25 | 0.50 | 0.75 | 1.00 | 1.25 | 1.50 | 1.75 | 2.00 | 2.25 | 2.50 | 2.75 | 3.00 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **Velocity (L/s)** | 0 | 0.22 | 0.45 | 0.61 | 0.75 | 0.82 | 0.85 | 0.83 | 0.74 | 0.61 | 0.43 | 0.23 | 0 |

**a)** Graph the data, and determine an equation that models the situation.
**b)** Use a graphing calculator to draw a scatter plot of the data. Enter your equation into the equation editor, and graph. Comment on the closeness of fit between the scatter plot and the graph.
**c)** At $t = 6$, what is the velocity of a typical person's breathing?
**d)** Estimate when the rate of change in the velocity of a person's breathing is the smallest during the first 3 s.
**e)** What is the significance of the value you found in part d)?
**f)** Estimate when the rate of change in the velocity of a person's breathing is the greatest during the first 3 s.

**8.** The following table shows the average number of monthly hours of sunshine for Toronto.

| Month | J | F | M | A | M | J | J | A | S | O | N | D |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Average Monthly Sunshine (h) | 95.5 | 112.6 | 150.5 | 187.7 | 229.7 | 254.9 | 278.0 | 244.0 | 184.7 | 145.7 | 82.3 | 72.6 |

*Source:* Environment Canada

**a)** Create a scatter plot of the number of hours of sunshine versus time, where $t = 1$ represents January, $t = 2$ represents February, and so on.

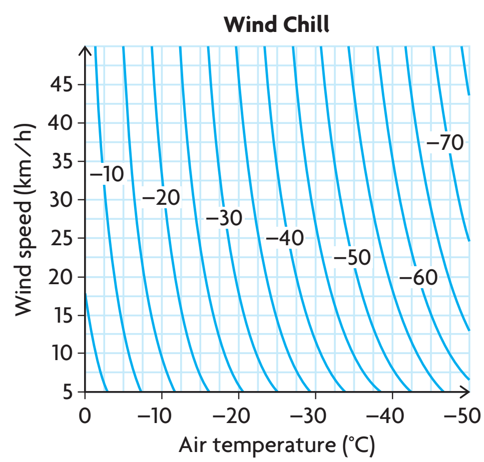
**b)** Draw the curve of best fit.

**c)** Determine a function that models this situation.

**d)** When will the number of monthly hours of sunshine be at a maximum according to the function? When will it be a minimum according to the function?

**e)** Discuss how well the model fits the data.

**9.** **T** The wind chill index measures the sensation of cold on the human skin. In October 2001, Environment Canada introduced the wind chill index shown. Each curve represents the combination of air temperature and wind speed that would produce the given wind chill value.




The following table gives the wind chill values when the temperature is $-20\,°C$.

| Wind Speed (km/h) | 5 | 10 | 15 | 20 | 25 | 30 | 35 | 40 | 45 | 50 | 55 | 60 | 65 | 70 | 75 | 80 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Wind Chill (°C) | −24 | −27 | −29 | −31 | −32 | −33 | −33 | −34 | −35 | −35 | −36 | −37 | −37 | −37 | −38 | −38 |

*Source:* Environment Canada

**a)** Create a graphical model for the data.

**b)** Determine an algebraic model for the data.

**c)** Use your model from part b) to predict the wind chill for a wind speed of 0 km/h, 100 km/h, and 200 km/h (hurricane force winds). Comment on the reasonableness of each answer.

**10.** The population of Canada is measured on a regular basis by taking a census. The table shows the population of Canada at the end of each period. From 1851 to 1951, each period is a 10-year interval. From 1951 to 2006, each period is a five-year interval.

**A**

| Period | Census Population at the End of a Period (in thousands) | Period | Census Population at the End of a Period (in thousands) |
|---|---|---|---|
| 1851–1861 | 3 230 | 1951–1956 | 16 081 |
| 1861–1871 | 3 689 | 1956–1961 | 18 238 |
| 1871–1881 | 4 325 | 1961–1966 | 20 015 |
| 1881–1891 | 4 833 | 1966–1971 | 21 568 |
| 1891–1901 | 5 371 | 1971–1976 | 23 450 |
| 1901–1911 | 7 207 | 1976–1981 | 24 820 |
| 1911–1921 | 8 788 | 1981–1986 | 26 101 |
| 1921–1931 | 10 377 | 1986–1991 | 28 031 |
| 1931–1941 | 11 507 | 1991–1996 | 29 672 |
| 1941–1951 | 13 648 | 1996–2001 | 30 755 |
| | | 2001–2006 | 31 613 |

*Source:* Statistics Canada, Demography Division

**a)** Use technology to investigate polynomial and exponential models for the relationship of the population and years since 1861. Describe how well each model fits the data.
**b)** Use each model to estimate Canada's population in 2016.
**c)** Which model gives the most realistic answer? Explain.
**d)** Use the model you chose in part c) to estimate the rate at which Canada's population was increasing in 2000.

**11.** The data shown model the growth of a rabbit population in an environment where the rabbits have no natural predators.
**a)** Determine an algebraic model for the data.
**b)** The original population of rabbits was 75; when does the model predict this was?
**c)** Discuss the growth rate of the rabbit population between 1955 and 1990.
**d)** Predict the rabbit population in 2020.

| Year | Rabbit Population |
|---|---|
| 1955 | 650 |
| 1958 | 2 180 |
| 1960 | 5 300 |
| 1961 | 8 200 |
| 1962 | 12 400 |
| 1965 | 35 500 |
| 1968 | 66 300 |
| 1975 | 91 600 |
| 1980 | 92 900 |
| 1986 | 92 800 |
| 1990 | 93 100 |

**12.** Household electrical power in North America is provided in the form of alternating current. Typically, the voltage cycles smoothly between $+155.6$ volts and $-155.6$ volts 60 times per second. Assume that at time zero the voltage is $+155.6$ volts.
**a)** Determine a sine function to model the alternating voltage.
**b)** Determine a cosine function to model the alternating voltage.
**c)** Which sinusoidal function was easier to determine? Explain.

| Time, $t$ (min) | Pressure, $P$ (kPa) |
|---|---|
| 0 | 400 |
| 5 | 335 |
| 10 | 295 |
| 15 | 255 |
| 20 | 225 |
| 25 | 195 |
| 30 | 170 |

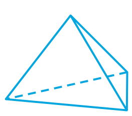
**13.** The pressure of a car tire with a slow leak is given in the table of values.
  **a)** Use technology to investigate linear, quadratic, and exponential models for the relationship of the tire pressure and time. Describe how well each model fits the data.
  **b)** Use each model to predict the pressure after 60 min.
  **c)** Which model gives the most realistic answer? Explain.

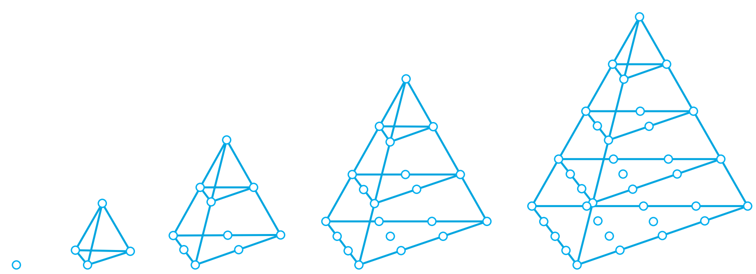
**14.** Explain why population growth is often exponential.

**15.** **C** Consider the various functions that could be used for mathematical models.
  **a)** Which functions could be used to model a situation in which the values of the dependent variable increase toward infinity? Explain.
  **b)** Which functions could be used to model a situation in which the values of the dependent variable decrease to zero? Explain.
  **c)** Which functions could be used to model a situation in which the values of the dependent variable approach a non-zero value? Explain.

## Extending

**16.** The numbers 1, 4, 10, 20, and 35 are called tetrahedral numbers because they are related to a four-sided shape called a tetrahedron.


tetrahedron


1 4 10 20 35

  **a)** Determine a mathematical model that you can use to generate the $n$th tetrahedral number.
  **b)** Is 47 850 a tetrahedral number? Justify your answer.

**17.** According to Statistics Canada, Canada's population reached 30.75 million on July 1, 2000—an increase of 256 700 from the previous year. The rate of growth for that year was the same as the rate of growth for the year before. Both Ontario and Alberta, however, recorded 1.3% growth rates in 2000.
  **a)** Create algebraic and graphical models for the population growth of Canada. Assume that the percent rate of growth was the same for every year.
  **b)** How does the growth rate for Canada's population compare with the growth rate reported by Ontario and Alberta?