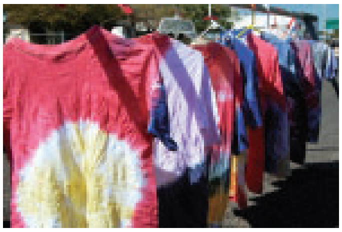
6.1 Solving Quadratic Equations

YOU WILL NEED

- grid paper
- ruler
- graphing calculator



GOAL

Use graphical and algebraic strategies to solve quadratic equations.

INVESTIGATE the Math

Andy and Susie run a custom T-shirt business. From past experience, they know that they can model their expected profit, in dollars, with the relation $P = -x^2 + 120x - 2000$, where x is the number of T-shirts they sell. Andy wants to sell enough T-shirts to earn $1200. Susie wants to sell just enough T-shirts to break even because she wants to close the business.

? How can Andy and Susie determine the number of T-shirts they must sell to achieve their goals?

quadratic equation

an equation that contains at least one term whose highest degree is 2; for example, $x^2 + x - 2 = 0$

A. Why can you use the **quadratic equation** $-x^2 + 120x - 2000 = 0$ to determine the number of T-shirts that must be sold to achieve Susie's goal?

B. Factor the left side of the equation in part A. Use the factors to determine the number of T-shirts that must be sold to achieve Susie's goal.

C. Use your factors for part B to predict what the graph of the profit relation will look like. Sketch the graph, based on your prediction.

D. Graph the profit relation using a graphing calculator. Was your prediction for part C correct?

E. What quadratic equation can you use to describe Andy's goal of making a profit of $1200?

root

a solution; a number that can be substituted for the variable to make the equation a true statement; for example, $x = 1$ is a root of $x^2 + x - 2 = 0$, since $1^2 + 1 - 2 = 0$

F. How can you use your graph for part D to determine the **roots** of your equation for part E?

G. How many T-shirts must be sold to achieve Andy's goal?

Reflecting

H. Why did factoring $-x^2 + 120x - 2000$ help you determine the break-even points?

I. Are the roots of the equation $-x^2 + 120x - 2000 = 0$ also zeros or x-intercepts of the relation $y = -x^2 + 120x - 2000$? Explain.

J. Why would factoring the left side of $-x^2 + 120x - 2000 = 1200$ not help you determine the number of T-shirts that Andy has to sell?

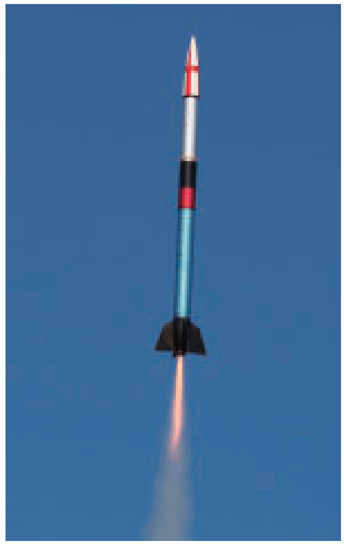
K. Explain why it would help you solve the equation in part J if you were to write it as $-x^2 + 120x - 2000 - 1200 = 0$.

L. To solve $ax + b = c$, you isolate x. Why would you not isolate x^2 to solve $ax^2 + bx + c = 0$?

APPLY the Math

EXAMPLE 1 Selecting a strategy to solve a quadratic equation

The user's manual for Arleen's model rocket says that the equation $h = -5t^2 + 40t$ models the approximate height, in metres, of the rocket after t seconds. When will Arleen's rocket reach a height of 60 m?



Amir's Solution: Selecting a factoring strategy

$$-5t^2 + 40t = 60$$

I substituted 60 for h because I wanted to calculate the time for the height 60 m.

$$-5t^2 + 40t - 60 = 0$$

I subtracted 60 from both sides of the equation to make the right side equal zero. I did this so that I could determine the zeros of the corresponding relation.

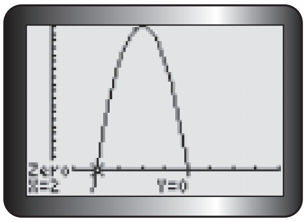
$$-5(t^2 - 8t + 12) = 0$$
$$-5(t - 2)(t - 6) = 0$$
$$t - 2 = 0 \text{ or } t - 6 = 0$$
$$t = 2 \qquad t = 6$$

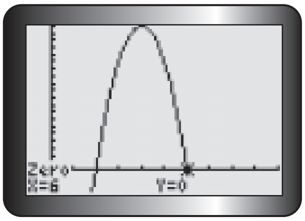
I divided out the common factor of -5. Then I factored the trinomial. The trinomial will equal zero if either factor equals zero. I set each factor equal to zero and solved both equations. This gave me the zeros of the parabola.

The rocket is 60 m above the ground at 2 s on the way up and 6 s on the way down.

Tech **Support**

For help locating the zeros of a relation using a TI 83/84 graphing calculator, see Appendix B-8. If you are using a TI-*n*spire, see Appendix B-44.



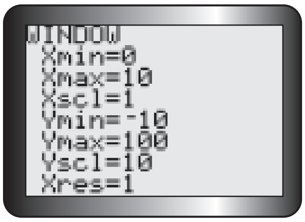


I verified my solutions by graphing $y = -5x^2 + 40x - 60$, and then locating the zeros.

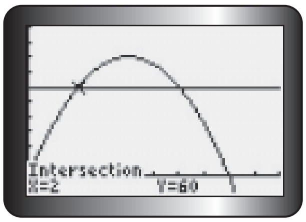
My solutions were correct.

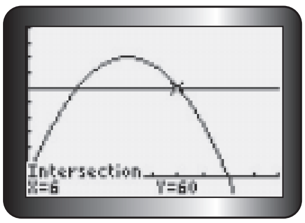
Alex's Solution: Selecting a graphing strategy

Using a graphing calculator, I entered the height equation in Y1. I substituted the variable *y* for *h* and the variable *x* for *t*. I entered 60 in Y2 to determine when the rocket will reach 60 m.



I estimated that the rocket will travel about 100 m and be in the air for about 10 s, so I used these window settings.





I used the Intersect operation to locate the intersection points of the two graphs.

Tech **Support**

For help determining points of intersection using a TI-83/84 graphing calculator, see Appendix B-11. If you are using a TI-*n*spire, see Appendix B-47.

The rocket is 60 m off the ground after 2 s and after 6 s.

EXAMPLE 2 Selecting a factoring strategy to solve a quadratic equation

Determine the roots of $6x^2 - 11x - 10 = 0$.

Annette's Solution

$6x^2 - 11x - 10 = 0$

Product $= -60$ Sum $= -11$

$(1)(-60)$ $1 + (-60) = -59$ ✘

$(2)(-30)$ $2 + (-30) = -28$ ✘

$(3)(-20)$ $3 + (-20) = -17$ ✘

$(4)(-15)$ $4 + (-15) = -11$ ✔

Since the trinomial in the equation contains no common factors and is one where $a \neq 1$, I used decomposition. I looked for two numbers whose sum is -11 and whose product is $(6)(-10) = -60$.

$$6x^2 - 15x + 4x - 10 = 0$$
$$3x(2x - 5) + 2(2x - 5) = 0$$
$$(2x - 5)(3x + 2) = 0$$

Since the numbers were -15 and 4, I used these to decompose the middle term. I factored the first two terms and then the last two terms. Then, I divided out the common factor of $2x - 5$.

$2x - 5 = 0$ or $3x + 2 = 0$

$2x = 5$ $3x = -2$

$x = \frac{5}{2}$ $x = -\frac{2}{3}$

I set each factor equal to zero and solved each equation.

The roots of $6x^2 - 11x - 10 = 0$ are $x = 2\frac{1}{2}$ and $x = -\frac{2}{3}$.

EXAMPLE 3 Reasoning about how to solve a quadratic equation

Determine all the values of x that satisfy the equation $x^2 + 4 = 3x(x - 5)$. If necessary, round your answers to two decimal places.

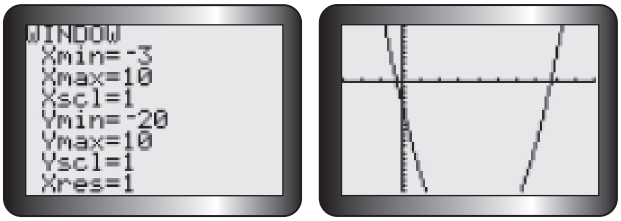
Karl's Solution

$$x^2 + 4 = 3x(x - 5)$$
$$x^2 + 4 = 3x^2 - 15x$$

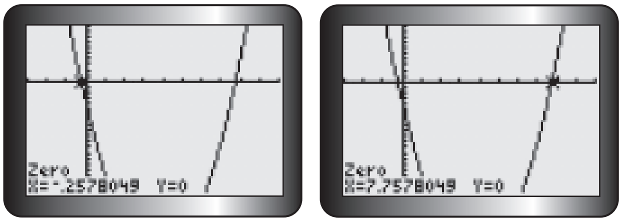
I decided to write an equivalent equation in the form $ax^2 + bx + c = 0$, which I could solve by graphing or factoring. I expanded the expression on the right side of the equation.

$$0 = 3x^2 - x^2 - 15x - 4$$
$$0 = 2x^2 - 15x - 4$$

I used inverse operations to make the left side of the equation equal to zero. I couldn't factor the right side of the equation, so I decided to use a graph.



I graphed $y = 2x^2 - 15x - 4$ using these window settings. From the graph, I could see that one x-intercept was between -1 and 0 and the other x-intercept was between 7 and 8.



Using the Zero operation of the calculator, I estimated that the x-intercepts were about -0.258 and 7.758.

Tech Support

For help determining the zeros of a relation using a TI-83/84 graphing calculator, see Appendix B-8. If you are using a TI-*n*spire, see Appendix B-44.

The solutions are $x \doteq -0.26$ and $x \doteq 7.76$.

I rounded the solutions to two decimal places. These are reasonable estimates, since the solutions are not exact.

EXAMPLE 4 Reflecting on the reasonableness of a solution

A ball is thrown from the top of a seaside cliff. Its height, h, in metres, above the sea after t seconds can be modelled by $h = -5t^2 + 21t + 120$. How long will the ball take to fall 20 m below its initial height?

Jacqueline's Solution

$h = -5t^2 + 21t + 120$
$h = -5(0)^2 + 21(0) + 120$
$h = 120$

I let $t = 0$ to determine the initial height of the ball.

The cliff is 120 m high, so the ball starts 120 m above the sea.

$120 - 20 = 100$

The initial height of the ball was 120 m. When the ball had fallen 20 m below its initial height, it was 100 m above the sea.

Let $h = 100$.
$100 = -5t^2 + 21t + 120$
$0 = -5t^2 + 21t + 120 - 100$
$0 = -5t^2 + 21t + 20$

I substituted 100 for h in the relation. I wrote the equation in the form $0 = ax^2 + bx + c$ so that I could solve it by graphing or factoring.

I subtracted 100 from both sides of the equation to make the left side equal to 0.

$$0 = 5t^2 - 21t - 20$$
$$0 = 5t^2 - 25t + 4t - 20$$
$$0 = 5t(t - 5) + 4(t - 5)$$
$$0 = (5t + 4)(t - 5)$$

I multiplied all the terms, on both sides of the equation, by -1 because I wanted $5t^2$ to be positive. I factored the right side of the equation using decomposition.

$5t + 4 = 0$ or $t - 5 = 0$

$5t = -4$ $\quad t = 5$

$t = -\frac{4}{5}$

I set each factor equal to zero and solved for t.

The ball will take 5 s to fall 20 m below its initial height.

Since the ball was thrown at $t = 0$, I knew that the solution $t = -\frac{4}{5}$ didn't make sense. I used the solution $t = 5$ since this did make sense.

In Summary

Key Ideas

- A quadratic equation is any equation that contains a polynomial in one variable whose degree is 2; for example, $x^2 + 6x + 9 = 0$.
- All quadratic equations can be expressed in the form $ax^2 + bx + c = 0$ using algebraic strategies. In this form, the equation can be solved by
  - factoring the quadratic expression, setting each factor equal to zero, and solving the resulting equations

    or
  - graphing the corresponding relation $y = ax^2 + bx + c$ and determining the zeros, or x-intercepts

Need to Know

- Roots and solutions have the same meaning. These are all values that satisfy an equation.
- Some quadratic equations can be solved by factoring. Other quadratic equations must be solved by using a graph.
- If you use factoring to solve a quadratic equation, write the equation in the form $ax^2 + bx + c = 0$ before you try to factor.
- To solve $ax^2 + bx + c = d$ using a graph, graph $y = ax^2 + bx + c$ and $y = d$ on the same axes. The solutions to the equation are the x-coordinates of the points where the parabola and the horizontal line intersect.

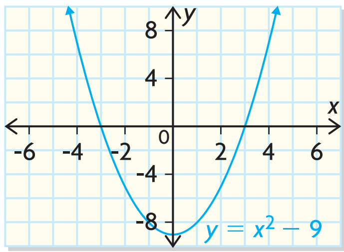
CHECK Your Understanding

1. The solutions to each equation are the x-intercepts of the corresponding quadratic relation. State the quadratic relation.

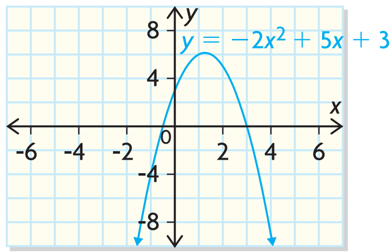
a) $x^2 - 4x + 4 = 0$ **b)** $2x^2 - 9x = 5$

2. Use the graph of each quadratic relation to determine the roots to each quadratic equation, where $y = 0$.

a)



b)



3. Solve each equation.

a) $x(x + 4) = 0$

b) $(x + 10)(x + 8) = 0$

c) $(x - 5)^2 = 0$

d) $(3x + 8)(x - 4) = 0$

e) $x^2 + 5x + 6 = 0$

f) $x^2 - 2x = 8$

PRACTISING

4. Determine whether the given value is a root of the equation.

a) $x = 2; x^2 + x - 6 = 0$

b) $x = 4; x^2 + 7x - 8 = 0$

c) $x = -\frac{1}{2}; 2x^2 + 11x + 5 = 0$

d) $x = \frac{3}{2}; 8x^2 + 10x - 3 = 0$

e) $x = -5; x^2 - 4x - 5 = 0$

f) $x = 2; 3x^2 - 2x - 8 = 0$

5. Solve each equation by factoring. Use an equivalent equation, if necessary.

a) $x^2 + 2x - 15 = 0$

b) $x^2 + 5x - 24 = 0$

c) $x^2 + 4x + 4 = 0$

d) $x^2 - 5x = 0$

e) $x^2 - 6x = 16$

f) $x^2 + 12 = 7x$

6. Solve by factoring. Verify your solutions.

a) $3x^2 - 5x - 2 = 0$

b) $2x^2 + 3x - 2 = 0$

c) $3x^2 - 4x - 15 = 0$

d) $6x^2 - x - 2 = 0$

e) $4x^2 - 4x = 3$

f) $9x^2 + 1 = 6x$

7. Simplify and then solve each equation.

K a) $x(x + 1) = 12$

b) $2x(x + 4) = x + 4$

c) $3x(x + 2) = 2x^2 - (4 - x)$

d) $3x(x + 6) + 50 = 2x^2 + 3(x - 2)$

e) $(x + 2)^2 + x = 2(3x + 5)$

f) $(2x + 1)^2 = x + 2$

8. Determine the roots of each equation.

a) $x^2 + 4x - 32 = 0$

b) $x^2 + 11x + 30 = 0$

c) $5x^2 - 28x - 12 = 0$

d) $x^2 + 5x = 14$

e) $4x^2 + 25 = 20x$

f) $3x^2 + 16x - 7 = 5$

9. Solve each equation. Round your answers to two decimal places.

a) $x^2 + 5x - 2 = 0$

b) $4x^2 - 8x + 3 = 0$

c) $x^2 + 1 = 4 - 2x^2$

d) $x(x + 5) = 2x + 7$

e) $3x^2 + 5x - 3 = x^2 + 4x + 1$

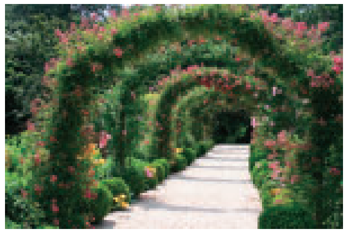
f) $(x + 3)^2 - 2x = 15$

10. Conor has a summer lawn-mowing business. Based on experience, Conor knows that $P = -5x^2 + 200x - 1500$ models his profit, P, in dollars, where x is the amount, in dollars, charged per lawn.

a) How much does he need to charge if he wants to break even?

b) How much does he need to charge if he wants to have a profit of $500?

11. Stacey maintains the gardens in the city parks. In the summer, she plans to build a walkway through the rose garden. The area of the walkway, A, in square metres, is given by $A = 160x + 4x^2$, where x is the width of the walkway in metres. If the area of the walkway must be 900 m^2, determine the width.



Environment *Connection*

By photosynthesis, green plants remove carbon dioxide from the air and produce oxygen.

12. Patrick owns an apartment building. He knows that the money he earns in a month depends on the rent he charges. This relationship can be modelled by $E = \frac{1}{50}R(1650 - R)$, where E is Patrick's monthly earnings, in dollars, and R is the amount of rent, in dollars, he charges each tenant.

a) How much will he earn if he sets the rent at $900?

b) If Patrick wants to earn at least $13 000, between what two values should he set the rent?

13. **T** Determine the points of intersection of the line $y = -2x + 7$ and the parabola $y = 2x^2 + 3x - 5$.

14. **A** While hiking along the top of a cliff, Harlan knocked a pebble over the edge. The height, h, in metres, of the pebble above the ground after t seconds is modelled by $h = -5t^2 - 4t + 120$.

a) How long will the pebble take to hit the ground?

b) For how long is the height of the pebble greater than 95 m?

15. **C** Is it possible to solve a quadratic equation that is not factorable over the set of integers? Explain.

16. a) Describe when and why you would rewrite a quadratic equation to solve it. In your answer, include $x^2 - 2x = 15$, rewritten as $x^2 - 2x - 15 = 0$.

b) Explain how the relation $y = x^2 - 2x - 15$ can be used to solve $x^2 - 2x - 15 = 0$.

Extending

17. Solve the equations $x^4 - 9x^2 + 20 = 0$ and $x^3 - 9x^2 + 20x = 0$ by first solving the equation $x^2 - 9x + 20 = 0$.

18. Will all quadratic equations always have two solutions? Explain how you know and support your claim with examples.