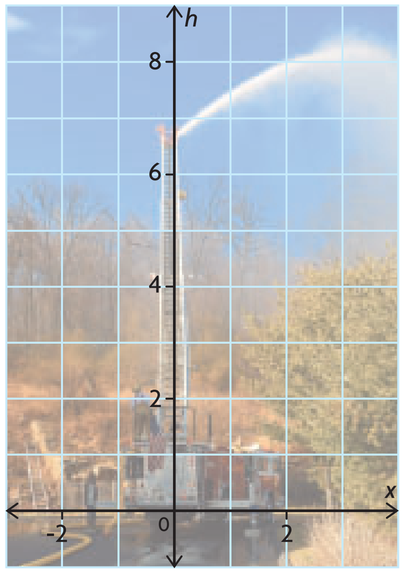
6.3 Completing the Square

GOAL

Write the equation of a parabola in vertex form by completing the square.

YOU WILL NEED

- algebra tiles (optional)
- grid paper
- ruler



LEARN ABOUT the Math

The automated hose on an aerial ladder sprays water on a forest fire. The height of the water, h, in metres, can be modelled by the relation $h = -2.25x^2 + 4.5x + 6.75$, where x is the horizontal distance, in metres, of the water from the nozzle of the hose.

? How high did the water spray from the hose?

Career *Connection*

As well as fighting fires, firefighters are trained to respond to medical and accident emergencies.

EXAMPLE 1 **Selecting a strategy** to solve a problem

Write the height relation $h = -2.25x^2 + 4.5x + 6.75$ in vertex form by **completing the square** to determine the maximum height.

completing the square

a process used to rewrite a quadratic relation that is in standard form, $y = ax^2 + bx + c$, in its equivalent vertex form, $y = a(x - h)^2 + k$

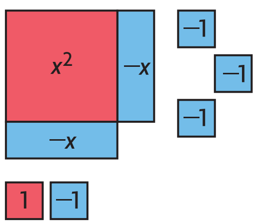
Joan's Solution: Selecting algebra tiles to complete the square

$$y = ax^2 + bx + c$$
$$y = a(x - h)^2 + k$$

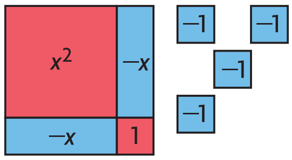
I knew that the vertex and maximum value can be determined from an equation in vertex form. I also knew that the value of a is the same in both standard form and vertex form.

$$h = -2.25x^2 + 4.5x + 6.75$$
$$h = -2.25(x^2 - 2x - 3)$$

I factored out -2.25 to get a trinomial with integer coefficients that I might be able to factor.



Because I wanted the equation in vertex form, I tried to make $x^2 - 2x - 3$ into a perfect square using tiles. I needed 1 positive unit tile in the corner to create a perfect square. I had 3 negative unit tiles, so I added 1 zero pair.



$$h = -2.25\left[(x - 1)^2 - 4\right]$$

I arranged the tiles to make the perfect square $x^2 - 2x + 1$. I had four negative unit tiles left over. This showed that $x^2 - 2x - 3 = (x - 1)^2 - 4$.

$$h = -2.25\left[(x - 1)^2 - 4\right]$$
$$h = -2.25(x - 1)^2 - (-2.25)(4)$$
$$h = -2.25(x - 1)^2 + 9$$

To write the equation in vertex form, I multiplied by -2.25 using the distributive property.

The water sprayed to a maximum height of 9 m above the ground.

The vertex of the parabola is (1, 9), and $a < 0$. So, the y-coordinate of the vertex gives the maximum height of the water.

Arianna's Solution: Selecting an algebraic strategy to complete the square

$$h = -2.25x^2 + 4.5x + 6.75$$
$$h = -2.25(x^2 - 2x) + 6.75$$

Since $a < 0$, the parabola opens downward and the maximum height is the y-coordinate of the vertex. I had to write the equation in vertex form.
To do so, I needed to create a perfect-square trinomial that used the variable x. I started by factoring out the coefficient of x^2 from the first two terms, since a perfect square can be created using the x^2 and x terms.

$$-\frac{2}{2} = -1 \text{ and } (-1)^2 = 1,$$
$$\text{so } x^2 - 2x + 1 = (x - 1)^2$$

To create a perfect square, the constant term had to be the square of half the coefficient of the x term.

$$h = -2.25(x^2 - 2x + 1 - 1) + 6.75$$

I knew that if I added 1 in the brackets, I would have to subtract 1 so that I did not change the equation.

$$h = -2.25[(x^2 - 2x + 1) - 1] + 6.75$$
$$h = -2.25[(x - 1)^2 - 1] + 6.75$$

I factored the perfect square.

$$h = -2.25(x - 1)^2 - (-2.25)(1) + 6.75$$
$$h = -2.25(x - 1)^2 + 9$$

I multiplied by −2.25 and collected like terms.

The vertex is (1, 9), so the water sprayed to a maximum height of 9 m above the ground.

Reflecting

A. Why did both Joan and Arianna factor out –2.25 first?

B. Whose strategy do you prefer? Why?

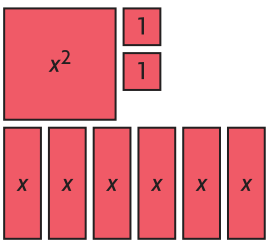
C. Explain how both strategies involve completing a square.

APPLY the Math

EXAMPLE 2 **Connecting** a model to the algebraic process of completing the square

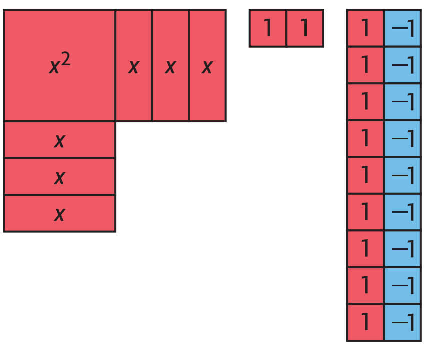
Write $y = x^2 + 6x + 2$ in vertex form, and then graph the relation.

Anya's Solution



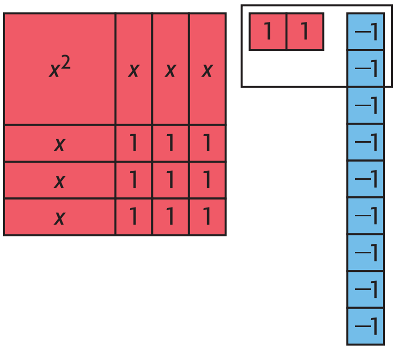
$$y = x^2 + 6x + 2$$

To write the relation in vertex form, I decided to complete the square using algebra tiles. Since there was only one x^2 tile, I had to make only one square.



$$y = x^2 + 6x + 9 - 9 + 2$$

I tried to form a square from the tiles, but I didn't have enough unit tiles. I needed 9 positive unit tiles to complete the square. To keep everything balanced, I added 9 zero pairs.



I completed the square. I had $(x + 3)$ for its length, with 2 positive unit tiles and 9 negative unit tiles left over. Since I could form 2 zero pairs, I had 7 negative unit tiles left over.

$y = (x^2 + 6x + 9) - 9 + 2$
$y = (x + 3)^2 - 9 + 2$
$y = (x + 3)^2 - 7$

$y = x^2 + 6x + 2$ in vertex form is
$y = (x + 3)^2 - 7$.

From the algebra tile model, I was able to write the relation in vertex form.

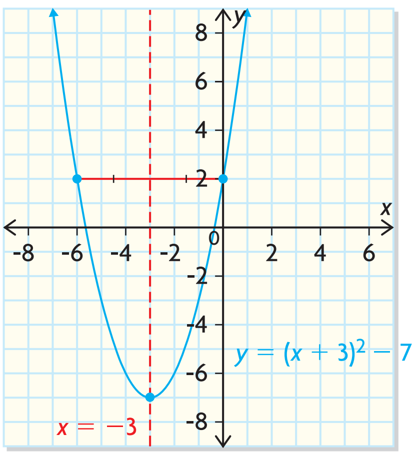
The vertex is $(-3, -7)$.
Since $a > 0$, the parabola opens upward.
The equation of the axis of symmetry is $x = -3$.

Using the vertex form of the equation, I determined the vertex, the direction of opening, and the equation of the axis of symmetry.

$y = (0 + 3)^2 - 7$
$y = 9 - 7$
$y = 2$

I let $x = 0$ to determine the y-intercept.

The y-intercept is 2.



I plotted the vertex and the y-intercept. I used symmetry to determine that $(-6, 2)$ is also a point on the parabola.

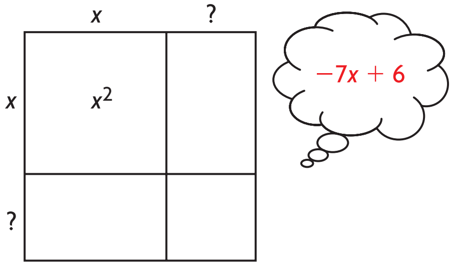
EXAMPLE 3 **Solving a problem** with an area diagram to complete the square

Cassidy's diving platform is 6 ft above the water. One of her dives can be modelled by the equation $d = x^2 - 7x + 6$, where d is her position relative to the surface of the water and x is her horizontal distance from the platform. Both distances are measured in feet. How deep did Cassidy go before coming back up to the surface?

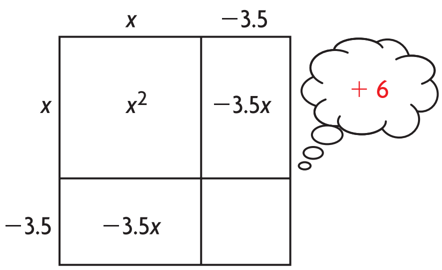
Sefu's Solution: Using an area diagram

$d = x^2 - 7x + 6$

Since the relation is quadratic and $a > 0$, Cassidy's deepest point will be at the vertex of a parabola that opens upward. I decided to complete the square.

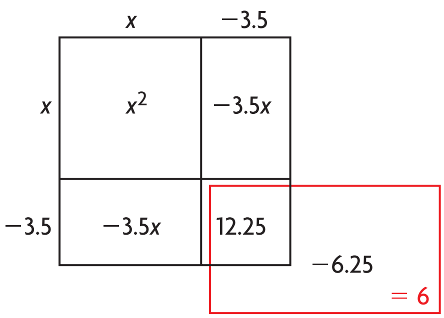


To complete the square, I drew a square area diagram. I knew that the length and width would have to be the same and that they both would be x plus a positive or negative constant.



Since the middle term $-7x$ had to be split equally between the two sides of the square, I found the constant by dividing the coefficient of the middle term by 2.

$$\frac{-7}{2} = -3.5$$



The constant term in the original equation is 6, but when I multiplied the -3.5s together to create the perfect square, I got 12.25, so I had to subtract 6.25.

$d = x^2 - 7x + 6$

$d = (x - 3.5)^2 - 6.25$

I used the dimensions of my square along with the extra -6.25 to write the equivalent relation in vertex form.

Cassidy dove to a depth of 6.25 ft before turning back toward the surface.

The vertex is $(3.5, -6.25)$ and since $a > 0$, the y-coordinate of the vertex is her lowest point.

EXAMPLE 4 | **Solving a problem** by determining the maximum value

Christopher threw a football. Its height, h, in metres, after t seconds can be modelled by $h = -4.9t^2 + 11.76t + 1.4$. What was the maximum height of the football, and when did it reach this height?

Macy's Solution

$h = -4.9t^2 + 11.76t + 1.4$
$h = -4.9(t^2 - 2.4t) + 1.4$

Since the relation is quadratic, the maximum value occurs at the vertex. To determine this value, I had to write the equation in vertex form. I started by factoring out -4.9 from the first two terms.

$\frac{-2.4}{2} = -1.2$ and $(-1.2)^2 = 1.44$

To determine the constant I had to add to $t^2 - 2.4t$ to create a perfect square, I divided the coefficient of t by 2. Then I squared my result.

$h = -4.9(t^2 - 2.4t + 1.44 - 1.44) + 1.4$
$h = -4.9[(t^2 - 2.4t + 1.44) - 1.44] + 1.4$
$h = -4.9[(t - 1.2)^2 - 1.44] + 1.4$

I completed the square by adding and subtracting 1.44, so the value of the expression value did not change. I grouped the three terms that formed the perfect square. Then I factored.

$h = -4.9(t - 1.2)^2 + 7.056 + 1.4$
$h = -4.9(t - 1.2)^2 + 8.456$
The vertex is (1.2, 8.456).

I multiplied by -4.9 using the distributive property. Then I added the constant terms.

The football reached a maximum height of 8.456 m after 1.2 s.

Since $a < 0$, the y-coordinate of the vertex is the maximum value. The x-coordinate is the time when the maximum value occurred.

In Summary

Key Idea

- A quadratic relation in standard form, $y = ax^2 + bx + c$, can be rewritten in its equivalent vertex form, $y = a(x - h)^2 + k$, by creating a perfect square within the expression and then factoring it. This technique is called completing the square.

Need to Know

- When completing the square, factor out the coefficient of x^2 from the terms that contain variables. Then divide the coefficient of the x term by 2 and square the result. This tells you what must be added and subtracted to create an equivalent expression that contains a perfect square.
- Completing the square can be used to determine the vertex of a quadratic relation in standard form.

CHECK Your Understanding

1. Copy and replace each symbol to complete the square.

a) $y = x^2 + 12x + 5$
$y = x^2 + 12x + \blacksquare - \blacksquare + 5$
$y = (x^2 + 12x + \blacksquare) - \blacksquare + 5$
$y = (x + \blacklozenge)^2 - \bullet$

b) $y = 4x^2 + 24x - 15$
$y = 4(x^2 + \blacksquare x) - 15$
$y = 4(x^2 + 6x + \blacklozenge - \blacklozenge) - 15$
$y = 4[(x^2 + 6x + \blacklozenge) - \blacklozenge] - 15$
$y = 4(x + \bullet)^2 - \text{⬟} - 15$
$y = 4(x + \bullet)^2 - \bigstar$

2. Write each relation in vertex form by completing the square.
a) $y = x^2 + 8x$ **b)** $y = x^2 - 12x - 3$ **c)** $y = x^2 + 8x + 6$

3. Complete the square to state the coordinates of the vertex of each relation.
a) $y = 2x^2 + 8x$ **b)** $y = -5x^2 - 20x + 6$ **c)** $y = 4x^2 - 10x + 1$

PRACTISING

4. Consider the relation $y = -2x^2 + 12x - 11$.
a) Complete the square to write the relation in vertex form.
b) Graph the relation.

5. Determine the maximum or minimum value of each relation by completing the square.
a) $y = x^2 + 14x$
b) $y = 8x^2 - 96x + 15$
c) $y = -12x^2 + 96x + 6$
d) $y = -10x^2 + 20x - 5$
e) $y = -4.9x^2 - 19.6x + 0.5$
f) $y = 2.8x^2 - 33.6x + 3.1$

6. Complete the square to express each relation in vertex form.
K Then graph the relation.
a) $y = x^2 + 10x + 20$
b) $y = -x^2 + 6x - 1$
c) $y = 2x^2 + 4x - 2$
d) $y = -0.5x^2 - 3x + 4$

7. Complete the square to express each relation in vertex form. Then describe the transformations that must be applied to the graph of $y = x^2$ to graph the relation.
a) $y = x^2 - 8x + 4$
b) $y = x^2 + 12x + 36$
c) $y = 4x^2 + 16x + 36$
d) $y = -3x^2 + 12x - 6$
e) $y = 0.5x^2 - 4x - 8$
f) $y = 2x^2 - x + 3$

8. Joan kicked a soccer ball. The height of the ball, h, in metres, can be
A modelled by $h = -1.2x^2 + 6x$, where x is the horizontal distance, in metres, from where she kicked the ball.
a) What was the initial height of the ball when she kicked it? How do you know?
b) Complete the square to write the relation in vertex form.
c) State the vertex of the relation.
d) What does each coordinate of the vertex represent in this situation?
e) How far did Joan kick the ball?

Health *Connection*

An active lifestyle contributes to good physical and mental health.

9. Carly has just opened her own nail salon. Based on experience, she knows that her daily profit, P, in dollars, can be modelled by the relation $P = -15x^2 + 240x - 640$, where x is the number of clients per day. How many clients should she book each day to maximize her profit?

10. The cost, C, in dollars, to hire landscapers to weed and seed a local park can be modelled by $C = 6x^2 - 60x + 900$, where x is the number of landscapers hired to do the work. How many landscapers should be hired to minimize the cost?

$y = -2x^2 + 16x - 7$

$y = -2(x^2 + 8x) - 7$

$y = -2(x^2 + 8x + 64 - 64) - 7$

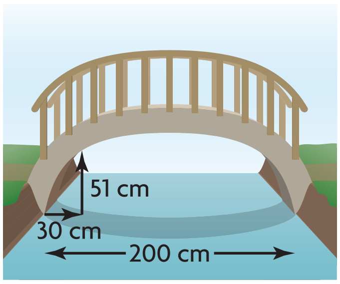
$y = -2(x + 8)^2 - 64 - 7$

$y = -2(x + 8)^2 - 73$

Therefore, the vertex is $(73, -8)$.

11. Neilles determined the vertex of a relation by completing the square, as shown at the left. When he checked his answer at the back of his textbook, it did not match the answer given. Identify each mistake that he made, explain why it is a mistake, and provide the correct solution.

12. **T** Bob wants to cut a wire that is 60 cm long into two pieces. Then he wants to make each piece into a square. Determine how the wire should be cut so that the total area of the two squares is as small as possible.



13. Kayli wants to build a parabolic bridge over a stream in her backyard as shown at the left. The bridge must span a width of 200 cm. It must be at least 51 cm high where it is 30 cm from the bank on each side. How high will her bridge be?

14. **C**
   a) Determine the vertex of the quadratic relation $y = 2x^2 - 4x + 5$ by completing the square.
   b) How does changing the value of the constant term in the relation in part a) affect the coordinates of the vertex?

15. The main character in a video game, Tammy, must swing on a vine to cross a river. If she grabs the vine at a point that is too low and swings within 80 cm of the surface of the river, a crocodile will come out of the river and catch her. From where she is standing on the riverbank, Tammy can reach a point on the vine where her height above the river, h, is modelled by the relation $h = 12x^2 - 76.8x + 198$, where x is the horizontal distance of her swing from her starting point. Should Tammy jump? Justify your answer.

16. Explain how to determine the vertex of $y = x^2 - 2x - 35$ using three different strategies. Which strategy do you prefer? Explain your choice.

Extending

17. Celeste has just started her own dog-grooming business. On the first day, she groomed four dogs for a profit of $26.80. On the second day, she groomed 15 dogs for a profit of $416.20. She thinks that she will maximize her profit if she grooms 11 dogs per day. Assuming that her profit can be modelled by a quadratic relation, calculate her maximum profit.

18. Complete the square to determine the vertex of $y = x^2 + bx + c$.