6.5 Interpreting Quadratic Equation Roots

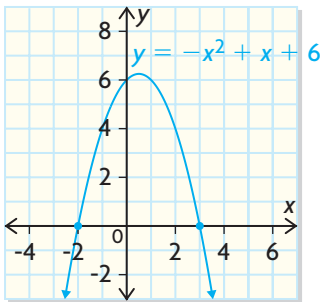### GOAL

Determine the number of roots of a quadratic equation, and relate these roots to the corresponding relation.

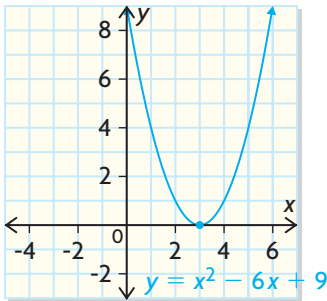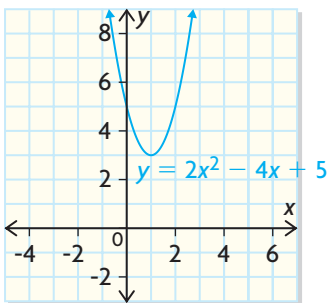## INVESTIGATE the Math

Quadratic relations may have two, one, or no x-intercepts.

This graph shows that the quadratic equation $-x^2 + x + 6 = 0$ has two solutions, $x = -2$ and $x = 3$.
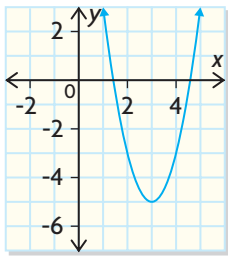

This graph shows that the quadratic equation $x^2 - 6x + 9 = 0$ has one solution, $x = 3$.

This graph shows that the quadratic equation $2x^2 - 4x + 5 = 0$ has no solutions.

? How can you determine the number of solutions to a quadratic equation without solving it?

A. Copy and complete this table using a graphing calculator. The first line has been completed for you.

Quadratic Relation	Sketch of Graph	Quadratic Equation Used to Determine x-intercepts	Roots of the Equation
$y = 2x^2 - 12x + 13$		$2x^2 - 12x + 13 = 0$	$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $x = \dfrac{-(-12) \pm \sqrt{(-12)^2 - 4(2)(13)}}{2(2)}$ $x = \dfrac{12 \pm \sqrt{40}}{4}$ $x \doteq 1.42 \text{ or } x \doteq 4.58$
$y = -2x^2 - 4x - 2$			
$y = -3x^2 + 9x + 12$			
$y = x^2 - 6x + 13$			
$y = -2x^2 - 4x - 5$			
$y = x^2 + 6x + 9$			

real numbers

the set of numbers that corresponds to each point on the number line shown; fractions, decimals, integers, and numbers like $\sqrt{2}$ are all real numbers
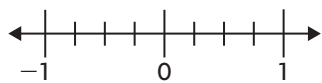
discriminant

the expression $b^2 - 4ac$ in the quadratic formula

B. How is the number of **real number** solutions related to the value of the **discriminant**?

Reflecting

C. Why are there no real number solutions when the discriminant is negative?

D. Why is there one real number solution when the discriminant is zero?

E. Why are there two real number solutions when the discriminant is positive?

APPLY the Math

EXAMPLE 1 | **Connecting** the real roots to the x-intercepts

Without solving, determine the number of real roots of each equation and describe the graph of the corresponding relation.
a) $3x^2 + 4x + 5 = 0$
b) $-2x^2 + 7x + 1 = 0$
c) $9x^2 - 12x + 4 = 0$

Steve's Solution

a) $3x^2 + 4x + 5 = 0$

$\begin{aligned} D &= b^2 - 4ac \\ &= 4^2 - 4(3)(5) \\ &= 16 - 60 \\ &= -44 \end{aligned}$

I substituted the values of a, b, and c into the discriminant, which I called D.

There are no real roots.

Since the discriminant is negative, there are no real roots.

The graph of $y = 3x^2 + 4x + 5$ has no x-intercepts. Since $a > 0$, the parabola opens upward and its vertex is above the x-axis.

b) $-2x^2 + 7x + 1 = 0$

$\begin{aligned} D &= b^2 - 4ac \\ &= 7^2 - 4(-2)(1) \\ &= 49 + 8 \\ &= 57 \end{aligned}$

I substituted the values of a, b, and c into the discriminant.

There are two real roots.

Since the discriminant is positive, there are two real roots.

The graph of $y = -2x^2 + 7x + 1$ has two x-intercepts. Since $a < 0$, the parabola opens downward and its vertex is above the x-axis.

c) $9x^2 - 12x + 4 = 0$

$\begin{aligned} D &= b^2 - 4ac \\ &= (-12)^2 - 4(9)(4) \\ &= 144 - 144 \\ &= 0 \end{aligned}$

I substituted the values of a, b, and c into the discriminant.

There is one real root.

Since the discriminant is zero, there is one real root.

The graph of $y = 9x^2 - 12x + 4$ has one x-intercept. Since $a > 0$, the parabola opens upward and its vertex is on the x-axis.

EXAMPLE **2**

Selecting a strategy to determine the number of zeros

Determine the number of zeros for $y = -2x^2 + 16x - 35$.

John's Solution: Completing the square

$y = -2x^2 + 16x - 35$
$y = -2(x^2 - 8x) - 35$
$y = -2(x^2 - 8x + 16 - 16) - 35$
$y = -2(x - 4)^2 + 32 - 35$
$y = -2(x - 4)^2 - 3$

> I completed the square to determine the vertex of the parabola. The vertex is at $(4, -3)$. Since $a < 0$, the parabola opens downward.

There are no zeros.

> Since the vertex is below the x-axis and the parabola opens downward, I knew that it could never cross the x-axis.

Erin's Solution: Using the quadratic formula

> The zeros occur when $y = 0$, so I substituted $y = 0$ into the relation. This resulted in a quadratic equation.

$y = -2x^2 + 16x - 35$
$0 = -2x^2 + 16x - 35$

$x = \dfrac{-16 \pm \sqrt{16^2 - 4(-2)(-35)}}{2(-2)}$

$x = \dfrac{-16 \pm \sqrt{-24}}{-4}$

> I substituted $a = -2$, $b = 16$, and $c = -35$ into the quadratic formula.
>
> I tried to calculate the square root of -24, but my calculator displayed this error message.

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ERR:NONREAL ANS
1∎Quit
2:Goto
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There are no real number solutions to the equation, so the relation has no x-intercepts.

> My calculator displayed an error message because there is no real number that can be multiplied by itself to give a negative result.

Cathy's Solution: Using the discriminant

$y = -2x^2 + 16x - 35$
$D = 16^2 - 4(-2)(-35)$
$\quad = 256 - 280$
$\quad = -24$

I substituted $a = -2$, $b = 16$, and $c = -35$ into the discriminant and evaluated.

The relation has no zeros.

Since the discriminant is negative, there are no zeros.

In Summary

Key Idea

- You can use the quadratic formula to determine whether a quadratic equation has two, one, or no real solutions, without solving the equation.

Need to Know

- The value of the expression $D = b^2 - 4ac$ gives the number of real solutions to a quadratic equation and the number of zeros in the graph of its corresponding relation.
  - If $D > 0$, there are two distinct real roots, or zeros.
  - If $D = 0$, there is one real root, or zero.
  - If $D < 0$, there are no real roots, or zeros.
- The direction of opening of a graph and the position of the vertex determines whether the graph has two, one, or no zeros and indicates whether the corresponding equation has two, one, or no real roots.

Two Real Roots	One Real Root	No Real Roots
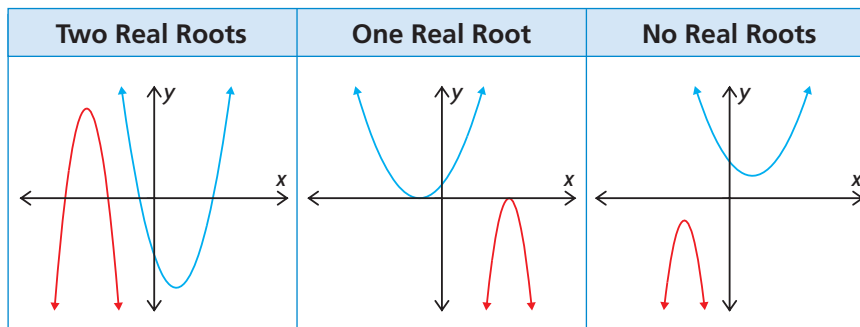
CHECK *Your Understanding*

1. **a)** Determine the roots of $x^2 - 6x + 5 = 0$ by using the quadratic formula and by factoring.
   b) What do your results for part a) tell you about the graph of $y = x^2 - 6x + 5$?
   **c)** Verify your answer for part b) using the discriminant.

2. Determine the number of real solutions that each equation has, without solving the equation. Explain your reasoning.

a) $(x - 1)^2 + 3 = 0$

b) $-2(x - 5)^2 + 8 = 0$

c) $5(x + 3)^2 = 0$

PRACTISING

3. Use the discriminant to determine the number of real solutions that each equation has.

a) $x^2 + 3x - 5 = 0$

b) $6x^2 + 5x + 12 = 0$

c) $-x^2 + 8x = 12$

d) $-2x^2 + 8x - 8 = 0$

e) $3x^2 + 2x = 5x + 12$

f) $-17x - 9 = 4x^2 - 5x$

4. State the number of times that each relation passes through the *x*-axis. Justify your answer.

a) $y = 3(x + 2)^2 - 5$

b) $y = -2(x + 5)^2 - 8$

c) $y = 2(x - 7)^2$

d) $y = 5(x - 12)^2 + 81$

e) $y = -4.9x^2 + 5$

f) $y = -6x^2$

5. Without graphing, determine the number of zeros that each relation has.

K **a)** $y = 3(x + 2)^2 + 5$

b) $y = -2x^2 + 3x - 7$

c) $y = x(x - 7)$

d) $y = 5(x + 2)(x + 2)$

e) $y = 3x^2 + 6x - 8$

f) $y = -6x^2 + 9$

6. Emma sells her handmade jewellery at a local market. She has always

A sold her silver toe rings for $10 each, but she is thinking about raising the price. Emma knows that her weekly revenue, *r*, in dollars, is modelled by $r = 250 + 5n - 2n^2$, where *n* is the amount that she increases the price. Is it possible for Emma to earn $500 in revenue? Explain.

7. In some places, a suspension bridge is the only passage over a river. The height of one such bridge, *h*, in metres, above the riverbed can be modelled by $h = 0.005x^2 + 24$.

a) How many zeros do you expect the relation to have? Why?

b) If the area was flooded, how high could the water level rise before the bridge was no longer safe to use?

8. The height of a super ball, *h*, in metres, can be modelled by $h = -4.9t^2 + 10.78t + 1.071$, where *t* is the time in seconds since the ball was thrown.

a) How many zeros do you expect this relation to have? Why?

b) Verify your answer for part a) algebraically.

c) How many times do you think the ball will pass through a height of 5 m? 7 m? 9 m?

d) Verify your answers for part c) algebraically.

9. Determine whether the vertex of each parabola lies above, below, or on the x-axis. Explain how you know.
   a) $h = 2t^2 - 4t + 1.5$
   c) $h = 5t^2 - 30t + 45$
   b) $h = 0.5t^2 - 2t + 0.5$
   d) $h = 0.5t^2 - 4t + 7.75$

10. For what value(s) of k does the equation $y = 5x^2 + 6x + k$ have each number of roots?
    a) two real roots
    b) one real root
    c) no real roots

11. Meg went bungee jumping from the Bloukrans River bridge in South Africa last summer. During the free fall on her first jump, her height above the water, h, in metres, was modelled by $h = -5t^2 + t + 216$, where t is the time in seconds since she jumped.
    a) How high above the water is the platform from which she jumped?
    b) Show that if her hair just touches the water on her first jump, the corresponding quadratic equation has two solutions. Explain what the solutions mean.

12. In the relation $y = 4x^2 + 24x - 5$, for which values of y will the corresponding equation have no solutions?

13. **T** A tangent is a line that touches a circle at exactly one point. For what values of k will the line $y = x + k$ be tangent to the circle $x^2 + y^2 = 25$?

14. **C** Sasha claims that the discriminant of a quadratic relation will never be negative if the relation can be written in the form $y = a(x - r)(x - s)$. Do you agree or disagree? Explain.

15. a) Create a quadratic relation, in vertex form, that has two zeros. Then write your relation in standard form. Use the discriminant to verify that it has two zeros.
    b) Create a quadratic relation, in vertex form, that has one zero. Then write your relation in standard form. Use the discriminant to verify that it has one zero.
    c) Create a quadratic relation, in vertex form, that has no zeros. Then write your relation in standard form. Use the discriminant to verify that it has one zero.
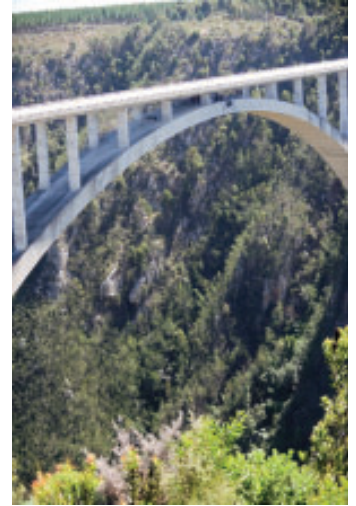
Extending

16. a) Write three quadratic equations in factored form.
    b) Expand and simplify your equations.
    c) Determine the discriminant for each equation.
    d) Explain how you can use the discriminant to determine whether an equation is factorable.

17. Determine the number of points of intersection of the relations $y = (x + 3)^2$ and $y = -2x^2 - 5$.