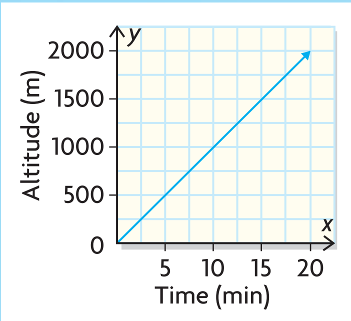
y
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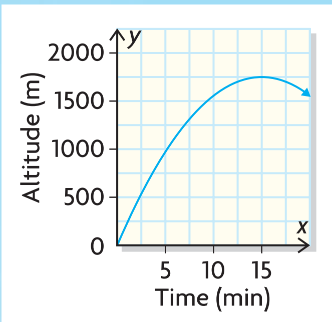
Altitude (m)
Time (min)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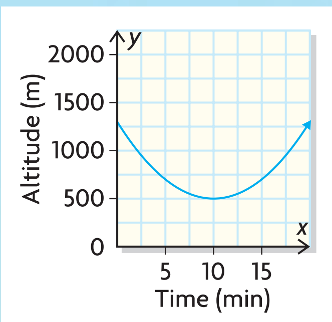
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Altitude (m)
Time (min)

Chapter

3

Graphs of Quadratic Relations

GOALS

You will be able to

- Describe the graphs and properties of quadratic relations of the forms $y = ax^2 + bx + c$ and $y = a(x - r)(x - s)$
- Expand and simplify quadratic expressions
- Apply quadratic models to solve problems

? What story does each graph tell about the movement of these balloons?

3 Getting Started

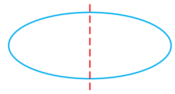
WORDS YOU NEED to Know

1. Match each term with the correct diagram or example.

   a) linear relation **c)** distributive property **e)** intercepts
   b) first differences **d)** scatter plot **f)** line of symmetry

i) $a(b + c) = ab + ac$

iii)

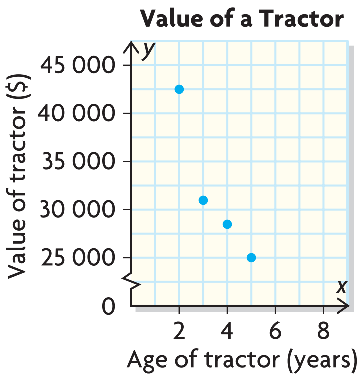


v) $y = 3x + 5$

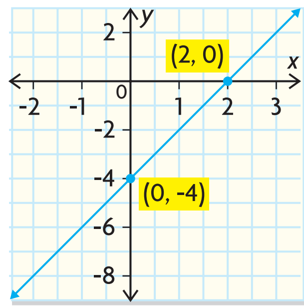
ii)

x	y	
1	12	
2	14	14 − 12 = 2
3	16	2
4	18	2
5	20	2

iv)



vi)



SKILLS AND CONCEPTS You Need

Curves as Mathematical Models

Sometimes, a curve is the best model for the relationship between the **dependent variable** and **independent variable** in a relation.

Study Aid

- For more help and practice, see Appendix A-11.

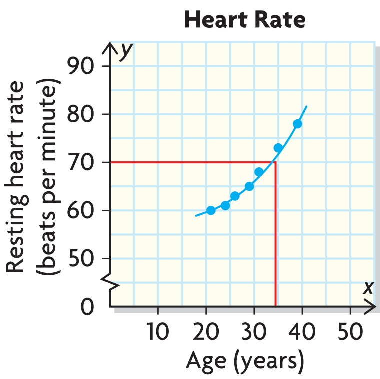
EXAMPLE

The resting heart rates of people of different ages are listed in the table.

Estimate the age of a person with a resting heart rate of 70 beats per minute.

Age (years)	Resting Heart Rate (beats per minute)
21	60
24	61
26	63
29	65
31	68
35	73
39	78

Solution



Create a scatter plot, and draw a curve. Interpolate by drawing a horizontal line from 70 on the *Resting heart rate* axis until it touches the curve. Draw a vertical line down to the *Age* axis. A person with a resting heart rate of 70 beats per minute is estimated to be about 34 years old.

2. Use the graph for resting heart rates to estimate

a) the resting heart rate of a person who is 15 years old

b) the age of a person with a resting heart rate of 75

3. This table shows the height of a baseball after it has been hit.

Time (s)	0	1	2	3	4	5
Height (m)	0.5	20.5	30.5	30.5	20.5	0.5

a) Create a scatter plot and draw a curve.

b) Estimate the height of the baseball at 2.5 s.

c) Estimate when the baseball will have a height of 25 m.

Multiplying a Polynomial by a Monomial

Several strategies can be used to multiply a **polynomial** by a **monomial**.

Study Aid

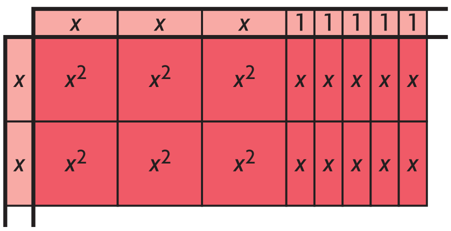
- For more help and practice, see Appendix A-8.

EXAMPLE

Multiply $2x(3x + 5)$.

Solution

Using an Algebra Tile Model



$2x(3x + 5) = 6x^2 + 10x$

Using the Distributive Property

$$\begin{aligned} 2x(3x + 5) &= 2x(3x) + 2x(5) \\ &= 6x^2 + 10x \end{aligned}$$

4. Expand and simplify each expression.

a) $4(x + 3)$

b) $2x(x - 5)$

c) $-3x^2(x - 2)$

d) $4x(2x - 3) + 3x(7 - 5x)$

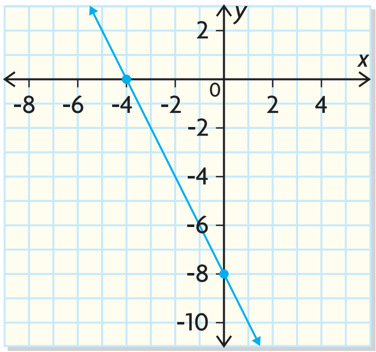
e) $7x^2(4x - 7 + 2x^2) - x(3x^2 - 5x - 2)$

f) $-4x(x^3 - 3x^2) + 2x(5x^2 - 3x) - 6x^3(x - 3)$

Study Aid

- For help, see the Review of Essential Skills and Knowledge Appendix.

Question	Appendix
5	A-5
6 to 9	A-7



PRACTICE

5. Determine the value of y for each given value of x.

a) $y = 2x - 3; x = 1.5$

b) $y = x^2; x = -3$

c) $y = x^2 + 2x - 1; x = 4$

d) $y = (2x + 1)(x - 3); x = 2$

6. A cell-phone plan costs \$25 each month plus \$0.10 per minute of airtime. Make a table of values, construct a graph, and write an equation to represent the monthly cost of the plan.

7. A laptop was purchased new for \$1000 and depreciates by \$200 each year. Make a table of values, construct a graph, and write an equation to represent the value of the laptop.

8. State the x- and y-intercepts for the relation in the graph at the left.

9. Determine the x- and y-intercepts for each relation. Then sketch the graph.

a) $y = 2x - 3$

b) $x + y = 5$

c) $y = 2x + 5$

d) $3x + 2y = 12$

e) $x = 2$

f) $y = -5$

10. State whether each statement is true or false. If the statement is false, create an example to support your decision.

a) Every table of values for a relation has constant first differences.

b) Every table of values for a linear relation has first differences equal to 0.

c) A relation can have no more than one y-intercept.

d) A relation can have no more than one x-intercept.

e) All 2-D figures have a line of symmetry.

11. Copy and complete the chart to show what you know about nonlinear relations.

Definition:

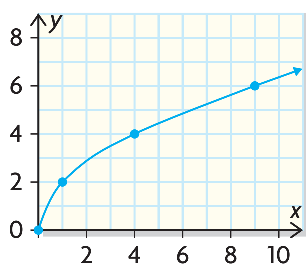
Characteristics:

Nonlinear Relations

Examples:

$C = \pi r^2$

x	0	1	2	3	4
y	1	3	9	27	81



Non-examples:

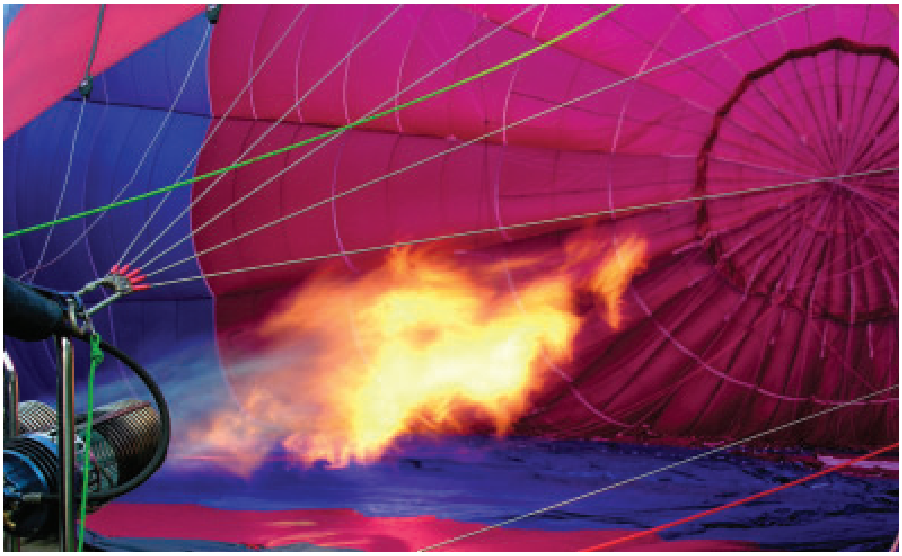
APPLYING What You Know

YOU WILL NEED

- grid paper
- ruler

Analyzing Balloon Data

Kajsa is a member of a ballooning club. She is responsible for analyzing the data collected by the club's training school on flights piloted by students. The following tables show data for two different training flights.



Training Flight 1

Time (s)	0	1	2	3	4	5	6
Height (m)	270	260	250	240	230	220	210

Training Flight 2

Time (s)	0	1	2	3	4	5	6
Height (m)	20	33	48	50	64	76	90

? How can you tell whether the data sets for the training flights are linear?

A. In which training flight is the balloon rising? In which training flight is the balloon descending? Explain how you know.

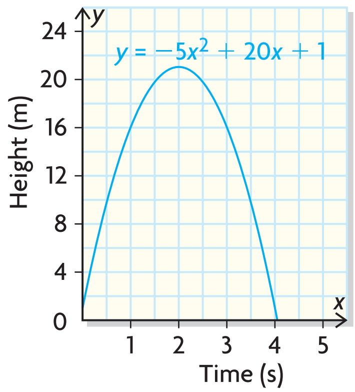
B. Describe three different strategies you could use to determine whether a data set is linear.

C. Use each strategy you described for part B to determine whether the data sets for the training flights are linear.

3.1 Exploring Quadratic Relations

YOU WILL NEED

- graphing calculator



quadratic relation in standard form

a relation of the form $y = ax^2 + bx + c$, where $a \neq 0$; for example, $y = 3x^2 + 4x - 2$

Tech Support

For help using a TI-83/84 graphing calculator to graph relations and create difference tables, see Appendix B-2 and B-7. If you are using a TI-*n*spire, see Appendix B-38 and B-43.

GOAL

Determine the properties of quadratic relations.

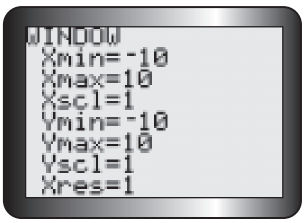
EXPLORE the Math

A "pop fly" in baseball occurs when the ball is hit straight up by the batter. For a certain pop fly, the height of the ball above the ground, y, in metres, is modelled by the relation $y = -5x^2 + 20x + 1$, where x is the time in seconds after the ball leaves the bat.

The path of the ball is straight up and down. The graph of the relation does not show this, however, because the ball rises fast at first and then more slowly due to gravity. The relation for the height of the baseball is an example of a **quadratic relation in standard form**.

? How does changing the coefficients and constant in $y = ax^2 + bx + c$ affect the graph of the quadratic relation?

A. Enter $y = x^2$ into a graphing calculator. Scroll to the left of Y1, and press ENTER to activate a thick line, then graph the relation by pressing GRAPH. Use the window settings shown.



Describe the shape of the graph and its symmetry.

B. Create a table of values for x from -4 to 4.

C. Calculate the **first differences** of the y-values. What do they confirm about the graph of the relation?

D. Calculate the **second differences**. What do you notice?

second differences

values that are calculated by subtracting consecutive first differences in a table of values

E. Investigate other relations of the form $y = ax^2$, where $a > 0$, $b = 0$, and $c = 0$. Repeat parts B to D for each relation in the graphing calculator screen at the right. Describe how the graphs and difference tables for these relations are the same and how they are different.

F. Investigate relations of the form $y = ax^2$, where $a < 0$, $b = 0$, and $c = 0$. Repeat parts B to D using these relations for Y2 to Y6: $y = -x^2$, $y = -2x^2$, $y = -5x^2$, $y = -0.5x^2$, and $y = -0.2x^2$. Describe how the **parabolas** and difference tables are the same and how they are different.

parabola

a symmetric graph of a quadratic relation, shaped like the letter "U" right-side up or upside down

G. Investigate relations of the form $y = ax^2 + c$, where $b = 0$. Repeat parts B to D using these relations for Y2 to Y5: $y = x^2 + 2$, $y = x^2 - 4$, $y = x^2 + 5$, and $y = x^2 - 6$. Make a conjecture to describe how changing the value of c affects a parabola.

H. Test your conjecture for part G by entering five new equations with the same value of a, but different values of c.

I. Investigate relations of the form $y = ax^2 + bx + c$. Repeat parts B to D using these relations for Y2 to Y5: $y = x^2 + 2x + 2$, $y = x^2 - 4x + 2$, $y = x^2 + 5x + 2$, and $y = x^2 - 6x + 2$. Make a conjecture to describe how changing the value of b affects a parabola.

J. Test your conjecture for part I by entering five new equations with the same value of a and the same value of c, but different values of b.

Reflecting

K. Describe what you noticed in the quadratic relations you examined about
- **i)** the **degree** of each equation
- **ii)** the second differences in the tables of values

L. Explain how the value of a in $y = ax^2 + bx + c$ relates to
- **i)** the graph of the parabola
- **ii)** the second differences in the table of values

M. Explain whether changing the value of b or c changes
- **i)** the location of the y-intercept of a parabola
- **ii)** the location of the line of symmetry of a parabola

In Summary

Key Ideas

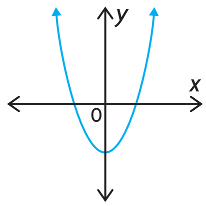
- The graph of any quadratic relation of the form $y = ax^2 + bx + c$, where $a \neq 0$, is a parabola that has a vertical line of symmetry.
- Any relation described by a polynomial of degree 2 is quadratic.

Need to Know

- For the quadratic relation $y = ax^2 + bx + c$,
  - the second differences are constant, but not zero
  - when the value of a (the coefficient of the x^2 term) is positive, the parabola opens upward and the second differences are positive
  - when the value of a (the coefficient of the x^2 term) is negative, the parabola opens downward and the second differences are negative
  - changing the value of b (the coefficient of the x term) changes the location of the line of symmetry of the parabola
  - the constant c is the value of the y-intercept of the parabola

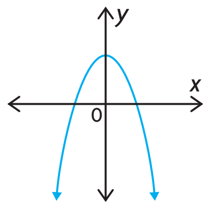
$y = ax^2 + bx + c$

$a > 0$



$y = ax^2 + bx + c$

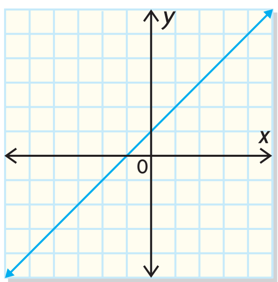
$a < 0$



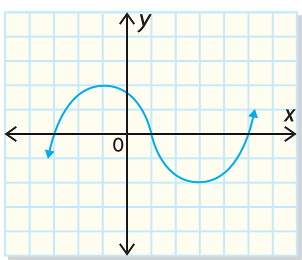
FURTHER Your Understanding

1. Which graphs appear to represent a quadratic relation? Explain.

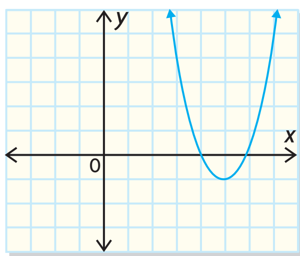
a)



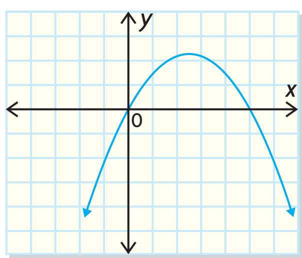
b)



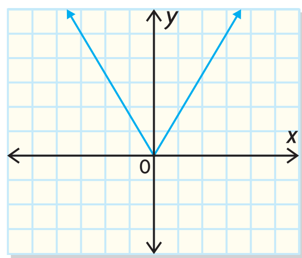
c)



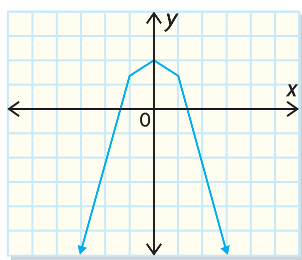
d)



e)



f)



2. a) Determine the degree of each relation.

i) $y = 5x - 2$

ii) $y = x^2 - 6x + 4$

iii) $y = x(x + 4)$

iv) $y = 2x^3 - 4x^2 + 5x - 1$

b) Which relations in part a) have a graph that is a parabola?

3. State the y-intercept of each quadratic relation in question 2.

4. Calculate the first differences for each set of data, and determine whether the relation is linear or nonlinear. If the relation is nonlinear, determine the second differences and identify the quadratic relations.

a)

x	10	20	30	40
y	21	41	61	81

b)

x	1	2	3	4
y	4	7	12	17

c)

x	5	6	7	8
y	−2	−3	−5	−8

d)

x	0	1	2	3
y	1	−1	7	−11

e)

x	0	1	2	3
y	−2	−1	6	25

f)

x	0	1	2	3	4
y	1	2	4	8	16

5. Each table of values represents a quadratic relation. Decide, without graphing, whether the parabola opens upward or downward.

a)

x	−3	−2	−1	0
y	2.5	5.0	6.5	7.0

b)

x	−2	−1	0	1	2
y	0	−5	0	15	40

c)

x	−2	−1	0	1	2
y	−3	3	5	3	−3

d)

x	0	1	2	3	4
y	−1	4	15	32	55

6. State whether the graph of each quadratic relation opens upward or downward. Explain how you know.

a) $y = x^2 - 1$

b) $y = -x^2 + 5x$

c) $y = -\frac{1}{2}x^2 + 6x - 4$

d) $y = -9x + \frac{1}{2}x^2 + 6$

7. Explain why the condition $a \neq 0$ must be stated to ensure that $y = ax^2 + bx + c$ is a quadratic relation.

3.2 Properties of Graphs of Quadratic Relations

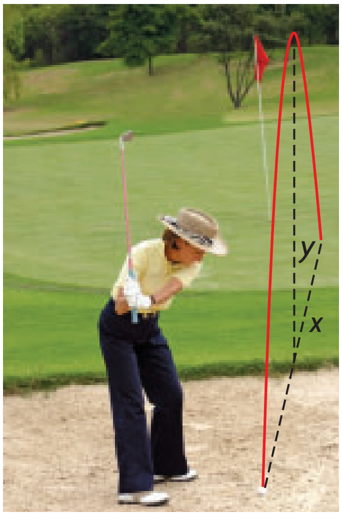
YOU WILL NEED

- grid paper
- ruler
- graphing calculator

GOAL

Describe the key features of the graphs of quadratic relations, and use the graphs to solve problems.

LEARN ABOUT the Math



Health *Connection*

Ultraviolet sun rays can damage the skin and cause skin cancer. Wearing a hat with a broad brim around the entire hat provides protection.

Grace hits a golf ball out of a sand trap, from a position that is level with the green. The path of the ball is approximated by the equation $y = -x^2 + 5x$, where x represents the horizontal distance travelled by the ball in metres and y represents the height of the ball in metres.

? What is the greatest height reached by the ball and how far away does it land?

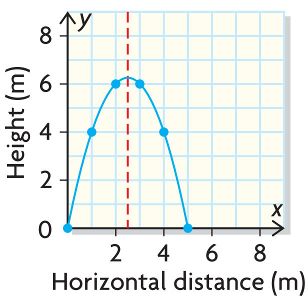
EXAMPLE 1 **Reasoning** from a table of values and a graph of a quadratic model

Determine the greatest height of the ball and the distance away that it lands.

Erika's Solution

x	0	1	2	3	4	5
y	0	4	6	6	4	0

I made a table of values. I used only positive values of x since the ball moves forward, not backward, when hit. When I reached a y-value of 0, I stopped. I assumed that the ball would not go below ground level.



I used my table of values to sketch the graph. I knew that the graph was a parabola, since the degree of the equation is 2. The parabola has a vertical line of symmetry that appears to pass through $x = 2.5$.

When $y = 0$, $x = \dfrac{0 + 5}{2} = 2.5$.

When $y = 4$, $x = \dfrac{1 + 4}{2} = 2.5$.

When $y = 6$, $x = \dfrac{2 + 3}{2} = 2.5$.

The equation of the axis of symmetry is $x = 2.5$.

I noticed that the points on the parabola with the same y-coordinate were the same distance from the line of symmetry. I reasoned that the **axis of symmetry** is the perpendicular bisector of any line segment joining points with the same y-coordinates. The means of the x-coordinates of these points give the equation of the axis of symmetry.

axis of symmetry
a line that separates a 2-D figure into two identical parts; if the figure is folded along this line, one of these parts fits exactly on the other part

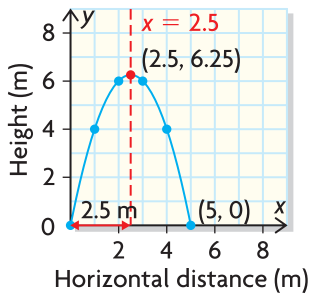
$y = -x^2 + 5x$
$y = -(2.5)^2 + 5(2.5)$
$y = -6.25 + 12.5$
$y = 6.25$

The coordinates of the vertex are (2.5, 6.25).

I saw that the **vertex** intersects the axis of symmetry, so its x-coordinate is 2.5. I substituted this value of x into the equation to get the **maximum value**.

vertex
the point of intersection of a parabola and its axis of symmetry

maximum value
the greatest value of the dependent variable in a relation



From my graph, I saw that $y = 0$ when $x = 5$. So, the ball lands 5 m away from where it was hit.

The ball's greatest height is 6.25 m. This occurs at a horizontal distance of 2.5 m from the starting point. The ball lands 5 m from the starting point.

Reflecting

A. Was the table of values or the graph more useful for determining the maximum height of the ball and the distance between where it was hit and where it landed? Explain.

B. How is the x-value at the maximum height of the ball related to the x-value of the point where the ball touches the ground?

C. Is it possible to predict whether a quadratic relation has a maximum value if you know the equation of the relation? Explain.

APPLY the Math

EXAMPLE 2 | **Selecting a table of values strategy** to graph a quadratic relation

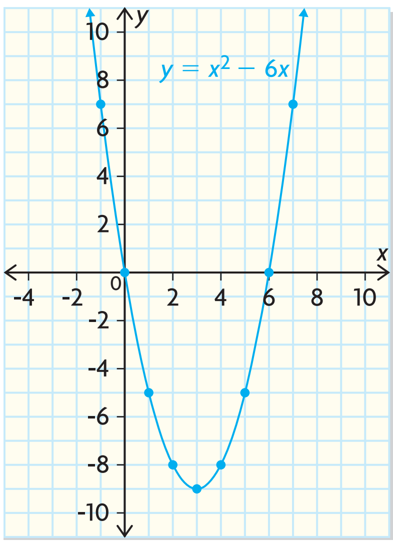
Sketch the graph of the relation $y = x^2 - 6x$. Determine the equation of the axis of symmetry, the coordinates of the vertex, the y-intercept, and the x-intercepts.

Cassandra's Solution

The relation $y = x^2 - 6x$ is quadratic.
$a = 1$, $b = -6$, and $c = 0$

The degree of the equation is 2, so the graph is a parabola. The coefficient of the x^2 term is $a = 1$. Since a is positive, the parabola opens upward.

x	y
−1	7
0	0
1	−5
2	−8
3	−9
4	−8
5	−5
6	0
7	7



I created a table of values using some negative and some positive x-values. I plotted each ordered pair, and drew a parabola that passed through each point. The parabola appears to have (3, −9) as its vertex and $x = 3$ as its axis of symmetry.

The points (1, −5) and (5, −5) are directly across from each other on the parabola.

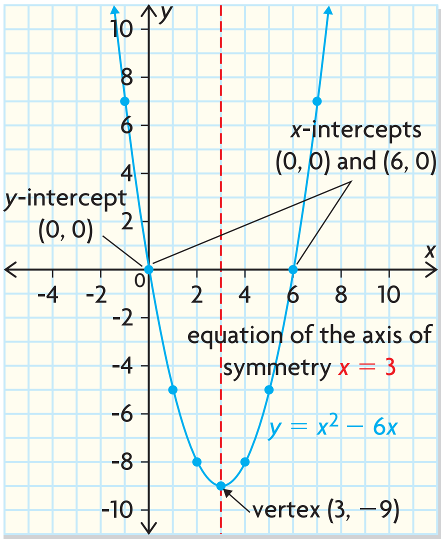
$$x = \frac{1 + 5}{2} \quad \text{so } x = 3$$

The equation of the axis of symmetry is $x = 3$.

I verified the equation of the axis of symmetry by averaging the x-coordinates of two points with the same y-value.

When $x = 3$ in $y = x^2 - 6x$,
$y = 3^2 - 6(3)$
$y = 9 - 18$
$y = -9$
The vertex occurs at (3, −9).

Since the vertex is on the axis of symmetry and the parabola, I substituted $x = 3$ into $y = x^2 - 6x$.



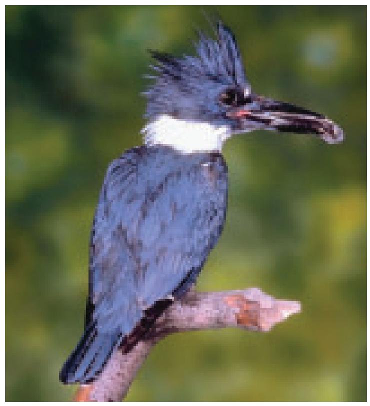
I looked at my graph to determine its features.

This parabola has $x = 3$ as the equation of its axis of symmetrry, the vertex is located at $(3, -9)$, the y-intercept is 0, and the x-intercepts are 0 and 6.

EXAMPLE 3 Selecting a strategy to determine the minimum value

A kingfisher dives into a lake. The underwater path of the bird is described by a parabola with the equation $y = 0.5x^2 - 3x$, where x is the horizontal position of the bird relative to its entry point and y is the depth of the bird underwater. Both measurements are in metres.

Graph the parabola. Use your graph to determine the equation of the axis of symmetry, the coordinates of the vertex, the y-intercept, and the x-intercepts. Calculate the bird's greatest depth below the water surface.



Pauline's Solution

Since $a = 0.5$, the parabola opens upward. The deepest point of the kingfisher's path is the **minimum value** of the relation. This occurs at the vertex of the parabola and corresponds to the y-coordinate of the vertex.

I made a plan to solve the problem.

Environment *Connection*

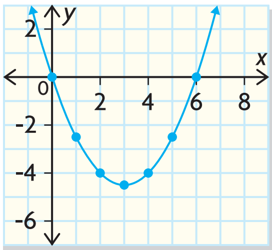
Since the Belted Kingfisher eats fish and crayfish, it is at risk due to toxins such as mercury.

minimum value

the least value of the dependent variable in a relation

x	y
0	0.0
1	−2.5
2	−4.0
3	−4.5
4	−4.0
5	−2.5
6	0.0

I created a table of values using the equation. I assumed that the bird moved from left to right, so I chose only positive values of x.



I plotted the points to create a graph; the vertex of the parabola appears to be (3, −4.5) and the equation of the axis of symmetry appears to be $x = 3$.

The y-intercept occurs at (0, 0). The x-intercepts occur at (0, 0) and (6, 0).

I looked at my graph to determine the intercepts.

$$x = \frac{0 + 6}{2}$$

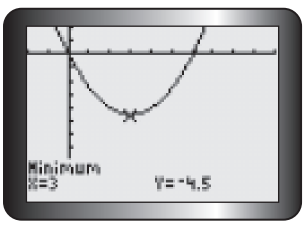
The equation of the axis of symmetry is $x = 3$.

The axis of symmetry is halfway between the x-intercepts, so I calculated the mean of the x-coordinates.

$$y = 0.5x^2 - 3x$$
$$y = 0.5(3)^2 - 3(3)$$
$$y = 4.5 - 9$$
$$y = -4.5$$

The vertex is on the axis of symmetry, so I substituted $x = 3$ into the equation to determine the y-coordinate.

The vertex is (3, −4.5).
The greatest depth of the bird, below the surface of the water, is 4.5 m.



I verified the minimum value of the relation using the minimum operation on a graphing calculator.

Tech Support

For help graphing a relation and determining its minimum value using a TI-83/84 graphing calculator, see Appendix B-9. If you are using a TI-*n*spire, see Appendix B-45.

EXAMPLE 4 Connecting a situation to a quadratic model

A model rocket is shot into the air from the roof of a building. Its height, h, above the ground, measured in metres, can be modelled by the equation $h = -5t^2 + 35t + 5$, where t is the time elapsed since liftoff in seconds.

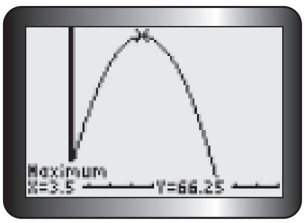
a) Determine the greatest height reached by the rocket.
b) How long is the rocket in flight?
c) Determine the height of the building.
d) When is the height of the rocket 61.25 m?

Liam's Solution

a) The equation is quadratic. Since $a = -5$, the graph is a parabola that opens downward. The greatest height occurs at the vertex. I entered the equation $y = -5x^2 + 35x + 5$ into a graphing calculator.

Since the calculator uses the variables x and y, I replaced the dependent variable h with y and the independent variable t with x.

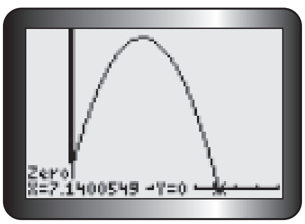
I adjusted the window settings until I could see the vertex. I used the maximum operation to determine the coordinates.



The vertex is (3.5, 66.25).

At 3.5 s after liftoff, the rocket reaches its greatest height of 66.25 m.

b)



The second zero (or x-intercept) corresponds to the rocket hitting the ground. I used the zero operation to determine the coordinates of this point.

The rocket is in flight for about 7.14 s.

Tech *Support*

For help graphing a relation, determining its maximum value, and determining its x-intercepts using a TI-83/84 graphing calculator, see Appendix B-2, B-9, and B-8. If you are using a TI-*n*spire, see Appendix B-38, B-45, and B-44.

Communication *Tip*

The zeros of a relation are its x-intercepts. "Zero" is another name for "x-intercept."

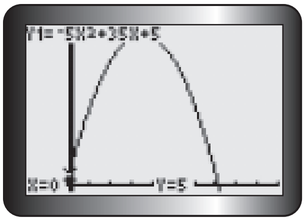
c) Let $x = 0$.

$y = -5(0)^2 + 35(0) + 5$

$y = 0 + 0 + 5$

$y = 5$

The height of the building corresponds to the *y*-intercept of the graph. This is the initial height of the ball, so I substituted $x = 0$ into the equation and solved for *y*.



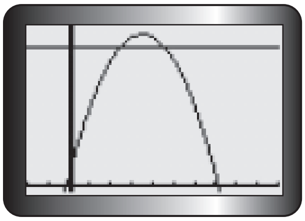
I verified my answer on a graphing calculator, using the value operation.

The building is 5.00 m tall.

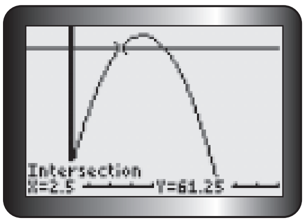
Tech Support

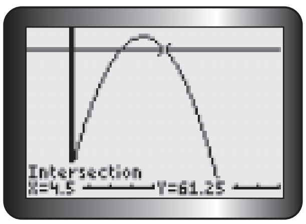
For help determining the value of a relation using a TI-83/84 graphing calculator, see Appendix B-3. If you are using a TI-*n*spire, see Appendix B-39.

d)



I had to determine when the height is 61.25 m. I entered the relation $y = 61.25$ into Y2 of the equation editor and re-graphed. The *x*-coordinates of the points of intersection of the horizontal line and the parabola tell me when this height occurs.





I used the intersect operation to determine the points of intersection. The first coordinate of each point represents a time when the rocket is 61.25 m above the ground.

Tech Support

For help determining the points of intersection for two relations using a TI-83/84 graphing calculator, see Appendix B-11. If you are using a TI-*n*spire, see Appendix B-47.

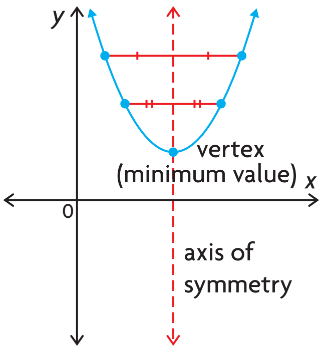
The rocket reaches a height of 61.25 m after 2.5 s on the way up and after 4.5 s on the way down.

In Summary

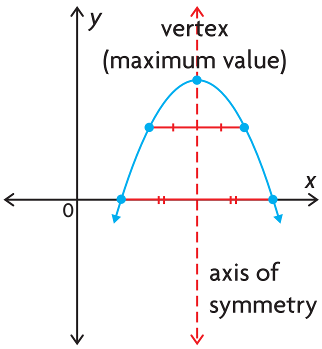
Key Ideas

- The vertex of a parabola with equation $y = ax^2 + bx + c$ is the point on the graph with
  - the least y-coordinate, or minimum value, if the parabola opens upward
  - the greatest y-coordinate, or maximum value, if the parabola opens downward

$y = ax^2 + bx + c, a > 0$



$y = ax^2 + bx + c, a < 0$



- A parabola with equation $y = ax^2 + bx + c$ is symmetrical with respect to a vertical line through its vertex. This line, or axis of symmetry, is the perpendicular bisector of any line segment that joins two points with the same y-coordinate on the parabola.

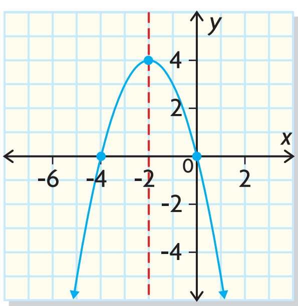
Need to Know

- The x-intercepts, or zeros, of a parabola can be determined by setting $y = 0$ in the equation of the parabola and solving for x.
- The y-intercept of a parabola can be determined by setting $x = 0$ in the equation of the parabola and solving for y.
- When a problem can be modelled by a quadratic relation, the graph of the relation can be used to estimate solutions to the problem.

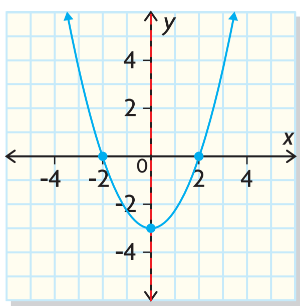
CHECK Your Understanding

1. For each graph, state the y-intercept, the zeros, the coordinates of the vertex, and the equation of the axis of symmetry.

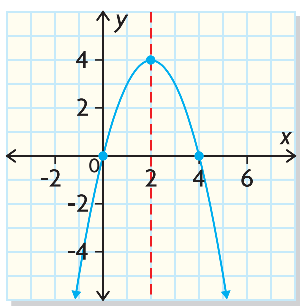
a)



b)



c)



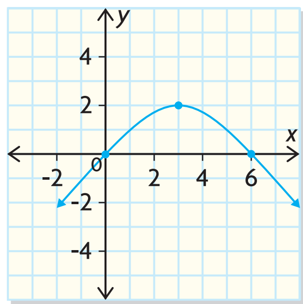
2. State the maximum or minimum value of each relation in question 1.

3. Two parabolas have the same x-intercepts, at (0, 0) and (10, 0). One parabola has a maximum value of 2. The other parabola has a minimum value of -4. Sketch the graphs of the parabolas on the same axes.

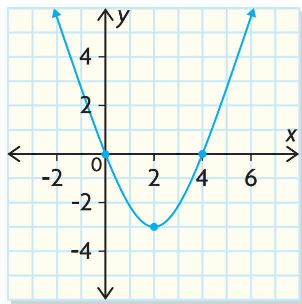
PRACTISING

4. Examine each parabola.
   i) Determine the coordinates of the vertex.
   ii) Determine the zeros.
   iii) Determine the equation of the axis of symmetry.
   iv) If you calculated the second differences, would they be positive or negative? Explain.

a)



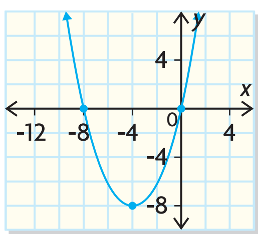
b)



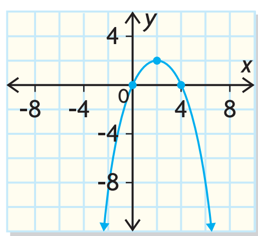
5. The zeros of a quadratic relation occur at $x = 0$ and $x = 6$. The second differences are positive.
   a) Is the y-value of the vertex a maximum value or a minimum value? Explain.
   b) Is the y-value of the vertex a positive number or a negative number? Explain.
   c) Determine the x-value of the vertex.

6. For each quadratic relation, state
   i) the equation of the axis of symmetry
   ii) the coordinates of the vertex
   iii) the y-intercept
   iv) the zeros
   v) the maximum or minimum value

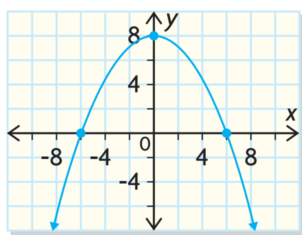
a)



b)



c)



7. Create a table of values for each quadratic relation, and sketch its graph. Then determine

- **i)** the equation of the axis of symmetry
- **ii)** the coordinates of the vertex
- **iii)** the y-intercept
- **iv)** the zeros
- **v)** the maximum or minimum value

a) $y = x^2 + 2$

b) $y = -x^2 - 1$

c) $y = x^2 - 2x$

d) $y = -x^2 + 4x$

e) $y = x^2 - 2x + 1$

f) $y = -x^2 - 2x + 3$

8. Use technology to graph each quadratic relation below. Then determine

- **i)** the equation of the axis of symmetry
- **ii)** the coordinates of the vertex
- **iii)** the y-intercept
- **iv)** the zeros
- **v)** the maximum or minimum value

a) $y = x^2 - 4x + 3$

b) $y = -x^2 + 4$

c) $y = x^2 + 6x + 8$

d) $y = -x^2 + 6x - 5$

e) $y = 2x(x - 4)$

f) $y = -0.5x(x - 8)$

9. Each pair of points is located on opposite sides of the same parabola. Determine the equation of the axis of symmetry for each parabola.

a) (3, 2), (9, 2)

b) (–18, 3), (7, 3)

c) (−5.25, −2.5), (3.75, −2.5)

d) $\left(-4\frac{1}{2}, 5\right), \left(-1\frac{1}{2}, 5\right)$

10. **K** Jen knows that (−1, 41) and (5, 41) lie on a parabola defined by the equation $y = 4x^2 - 16x + 21$. What are the coordinates of the vertex?

11. **C** State whether you agree or disagree with each statement. Explain why.

a) All quadratic relations of the form $y = ax^2 + bx + c$ have two zeros.

b) All quadratic relations of the form $y = ax^2 + bx + c$ have one y-intercept.

c) All parabolas that open downward have second differences that are positive.

Use a graphing calculator to answer questions 12 to 15.

12. A football is kicked into the air. Its height above the ground is approximated by the relation $h = 20t - 5t^2$, where h is the height in metres and t is the time in seconds since the football was kicked.

a) What are the zeros of the relation? When does the football hit the ground?

b) What are the coordinates of the vertex?

c) Use the information you found for parts a) and b) to graph the relation.

d) What is the maximum height reached by the football? After how many seconds does the maximum height occur?

13. A company that manufactures MP3 players uses the relation
A $P = 120x - 60x^2$ to model its profit. The variable x represents the number of thousands of MP3 players sold. The variable P represents the profit in thousands of dollars.

a) What is the maximum profit the company can earn?
b) How many MP3 players must be sold to earn this profit?
c) The company "breaks even" when the profit is zero. Are there any break-even points for this company? If so, how many MP3 players are sold at the break-even points?

Career Connection

Coast guard rescuers drop rafts that inflate within seconds to keep people afloat.

14. An inflatable raft is dropped from a hovering helicopter to a boat in distress below. The height of the raft above the water, in metres, is approximated by the equation $y = 500 - 5x^2$, where x is the time in seconds since the raft was dropped.

a) What is the height of the helicopter above the water?
b) When does the raft reach the water?
c) What is the height of the raft above the water 6 s after it is dropped?
d) When is the raft 100 m above the water?

15. Gamez Inc. makes handheld video game players. Last year,
T accountants modelled the company's profit using the equation $P = -5x^2 + 60x - 135$. This year, accountants used the equation $P = -7x^2 + 70x - 63$. In both equations, P is the profit, in hundreds of thousands of dollars, and x is the number of game players sold, in hundreds of thousands. If the same number of game players were sold in these years, did Gamez Inc.'s profit increase? Justify your answer.

16. a) Explain how the value of a in a quadratic relation, given in standard form, can be used to determine if the quadratic relation has a maximum value or a minimum value.
b) Explain how the coordinates of the vertex are related to the maximum or minimum value of the parabola.

Extending

17. a) Determine first and second differences for each relation.

i) $x = 2y^2$ **iii)** $y = x^3$
ii) $y = 2^x$ **iv)** $y = 2x^4$

b) Are graphs for any of the relations in part a) parabolas? Explain.
c) Are any of the relations in part a) quadratic? Explain.

18. The x-coordinate of the vertex of the graph of $y = 5x^2 - 3.2x + 8$ is $x = 0.32$. The number 0.32 is very similar to -3.2, which is the coefficient of x in the equation. Is this just a coincidence? Investigate several examples. Then make a conjecture and try to prove it.

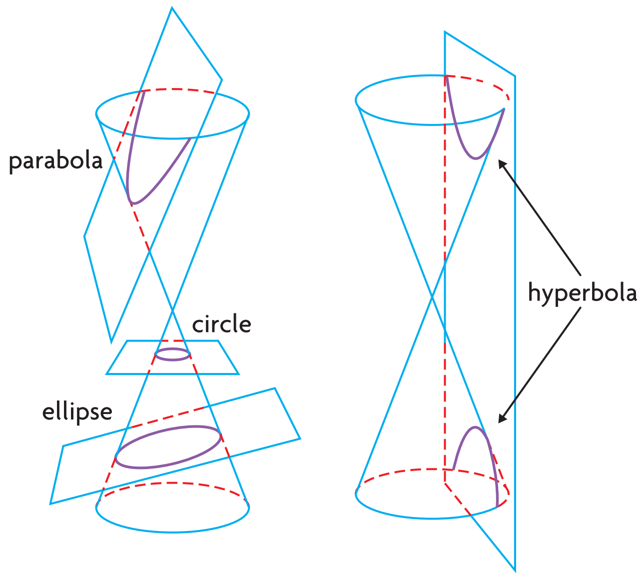
Curious | Math

Folding Paper to Create a Parabola

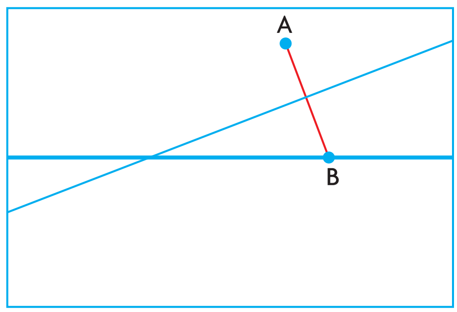
YOU WILL NEED
- uncreased paper or waxed paper
- dark marker
- ruler

People in early civilizations knew that the parabola was an important curve. In 350 BCE, Menaechmus, a student of Plato and Eudoxus, studied the curves that are formed when a plane intersects the surface of a cone. He discovered that there are several possibilities and that one of these possibilities is a parabola.

In 220 BCE, Apollonius named the curves shown. In 212 BCE, Archimedes studied the properties of these curves. One of the properties can be used to create a parabola by folding paper.



1. Begin with a clean, uncreased piece of paper.
2. With a ruler and a dark marker, draw a line across the midsection of the piece of paper.
3. Mark a point anywhere on the paper, except on the line. Label the point *A*.
4. Mark another point anywhere on the line, and label it *B*.
5. Using the ruler, draw line segment *AB*. Fold the paper so that point *A* lies directly on top of point *B*.



6. Crease the paper so that you can see the fold mark when the paper has been flattened out. To see the fold mark better, use the ruler and a pencil to draw a line along it. Make a conjecture about the relationship between line segment *AB* and the fold line.
7. Fold the paper so that point *A* falls on the original line but not on point *B*. Make a crease so that you can easily see the fold mark. Draw a line over the fold mark in pencil.
8. Repeat step 7 about 10 more times. Fold point *A* to a different point on the original line each time. Be sure to choose an equal number of points to the left and to the right of point *B*.
9. Describe the location of the parabola.

3.3 Factored Form of a Quadratic Relation

YOU WILL NEED

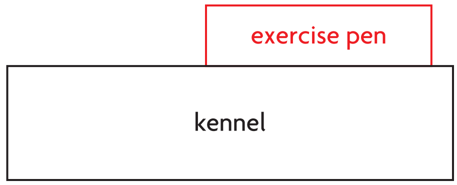
- grid paper and ruler, or graphing calculator

GOAL

Relate the factors of a quadratic relation to the key features of its graph.

INVESTIGATE the Math

Boris runs a dog kennel. He has purchased 80 m of fencing to build an outdoor exercise pen against the wall of the kennel.



Career *Connection*

Jobs at a dog kennel include kennel technician, veterinary technician, consultant, groomer, dog walker, and secretary.

? What dimensions should Boris use to maximize the area of the exercise pen?

A. If x represents the width of the pen, write an expression for its length.

B. Write a relation, in terms of x, for the area of the exercise pen. Identify the factors of the relation.

C. Create a table of values, and graph the relation you wrote for part B.

D. Use your table of values or graph to verify that the area relation is quadratic.

E. Does the relation have a maximum value or a minimum value? Explain how you know.

F. Determine the zeros of the parabola.

G. Determine the equation of the axis of symmetry of the parabola.

H. Determine the vertex of the parabola.

I. What are the dimensions that maximize the area of the exercise pen?

Reflecting

J. How are the factors of this relation related to the zeros of the graph?

K. The area relation can also be written as $A = -2(x)(x - 40)$ or $A = -2(x - 0)(x - 40)$, by dividing out the common factor of -2 from one of the factors. Explain why the **factored form of a quadratic relation** is useful when graphing the relation by hand.

factored form of a quadratic relation

a quadratic relation that is written in the form $y = a(x - r)(x - s)$

L. What is the area of the largest exercise pen that Boris can build?

APPLY the Math

EXAMPLE 1 Reasoning about the nature of a relation

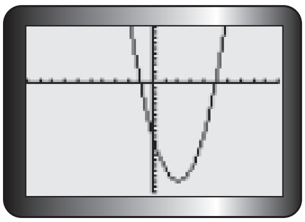
Is the graph of $y = 2(x + 1)(x - 5)$ a parabola? If so, in what direction does it open? Justify your answer.

Jasper's Solution

x	y	First Difference	Second Difference
−3	32		
		$14 - 32 = -18$	
−2	14		$-14 - (-18) = 4$
		−14	
−1	0		4
		−10	
0	−10		4
		−6	
1	−16		4
		−2	
2	−18		4
		2	
3	−16		

I created a table of values. Then I calculated the first and second differences. The second differences are constant but not zero, and they are also positive.

I predict that the graph of this relation is a parabola that opens upward.



I used a graphing calculator to graph the relation and check my predictions.

My predictions were correct.

EXAMPLE 2 Selecting a strategy to graph a quadratic relation given in factored form

Determine the y-intercept, zeros, axis of symmetry, and vertex of the quadratic relation $y = 2(x - 4)(x + 2)$. Then sketch the graph.

Cindy's Solution

$y = 2(x - 4)(x + 2)$
$y = 2(0 - 4)(0 + 2)$
$y = 2(-4)(2)$
$y = -16$

To determine the y-intercept, I substituted $x = 0$ into the equation. I noticed that multiplying the numbers in the original equation would have given the same result.

The y-intercept occurs at $(0, -16)$.

$0 = 2(x - 4)(x + 2)$
$x - 4 = 0$ or $x + 2 = 0$
$x = 4 \qquad x = -2$
The zeros occur at $(4, 0)$ and $(-2, 0)$.

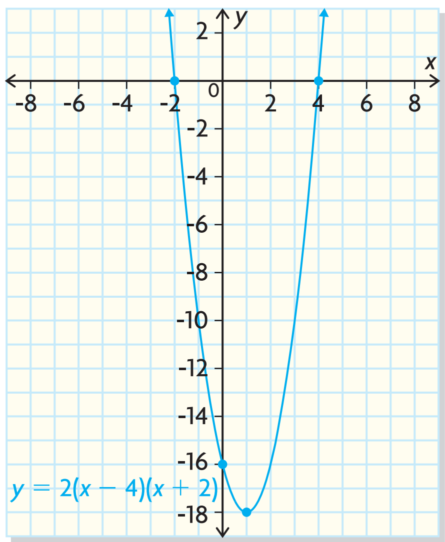
To determine the zeros, I let $y = 0$. I know that a product is zero only when one of its factors is zero, so I set each factor equal to 0 and solved for x.

$$x = \frac{4 + (-2)}{2}$$
$x = 1$
The equation of the axis of symmetry is $x = 1$.

The axis of symmetry passes through the midpoint of the zeros, so I calculated the mean.

$y = 2(x - 4)(x + 2)$
$y = 2(1 - 4)(1 + 2)$
$y = 2(-3)(3)$
$y = -18$
The vertex is $(1, -18)$.

The vertex lies on the axis of symmetry, so its x-coordinate is 1. I substituted $x = 1$ into the equation of the parabola to determine the y-coordinate.



I plotted the y-intercept, zeros, and vertex. Then I joined the points with a smooth curve.

EXAMPLE 3 Selecting a strategy to graph a quadratic relation given in factored form

Determine the y-intercept, zeros, axis of symmetry, and vertex of the quadratic relation $y = (x - 2)^2$. Then sketch the graph.

Kylie's Solution

$y = (x - 2)^2$
$y = (0 - 2)^2$
$y = 4$
The y-intercept occurs at $(0, 4)$.

To determine the y-intercept, I substituted $x = 0$ into the equation and solved for y.

$y = (x - 2)^2$
$0 = (x - 2)^2$
$0 = x - 2$
$x = 2$
The zero occurs at (2, 0).

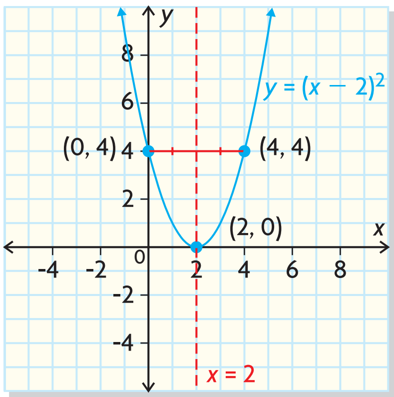
To determine the zeros, I let $y = 0$ and solved for x. Both factors are the same, since $y = (x - 2)^2$ is the same as $y = (x - 2)(x - 2)$. There is only one solution to $0 = x - 2$, so there is only one zero for the quadratic relation.

The equation of the axis of symmetry is $x = 2$.

The axis of symmetry passes through the midpoint of the zeros. Since there is only one zero, the axis of symmetry must pass through it.

The vertex is (2, 0).

Since (2, 0) is on the line $x = 2$, this point is also the vertex.

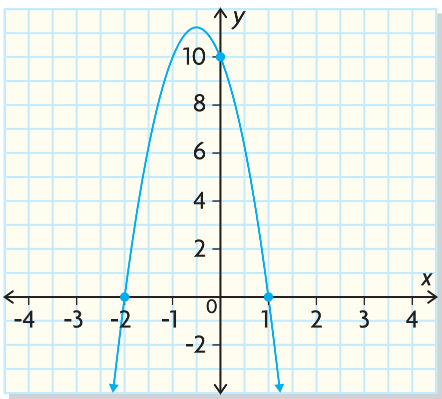


The y-intercept, (0, 4), is 2 units to the left of the axis of symmetry. There must be another point with y-coordinate 4 on the parabola, 2 units to the right of $x = 2$. This point is (4, 4).

I plotted these three points and joined them with a smooth curve.

EXAMPLE 4 Connecting the features of a parabola to its equation

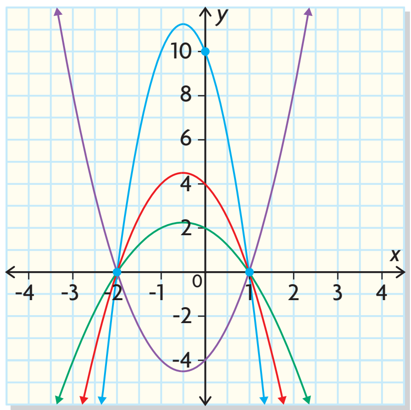
Determine an equation for this parabola.



Petra's Solution

The zeros occur at (−2, 0) and (1, 0).
$y = a(x - r)(x - s)$
$y = a[x - (-2)](x - 1)$
$y = a(x + 2)(x - 1)$

I determined the zeros of the parabola and substituted them into the factored form of a quadratic relation. I did this because I know that a parabola is described by a quadratic relation.



There are infinitely many parabolas with these zeros, all with different *y*-intercepts. A few examples are shown in the diagram. To determine the equation of the given parabola, I need to determine the value of *a*. This is the only value that varies in my equation $y = a(x + 2)(x - 1)$.

y-intercept occurs at (0, 10).

$$y = a(x + 2)(x - 1)$$
$$10 = a(0 + 2)(0 - 1)$$
$$10 = a(2)(-1)$$
$$10 = -2a$$
$$-5 = a$$

I chose the *y*-intercept to substitute into my equation because its coordinates are integers. Then I solved for *a*.

An equation for the given parabola is $y = -5(x + 2)(x - 1)$.

I substituted the value of *a* into my equation.

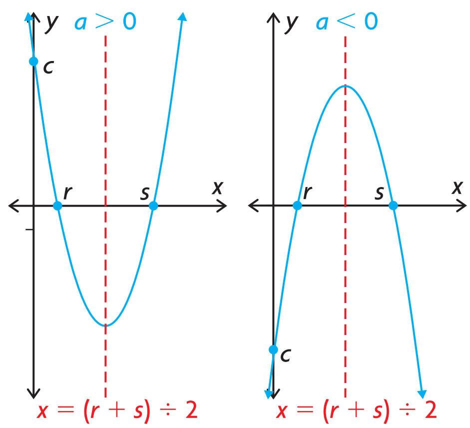
In Summary

Key Ideas

- When a quadratic relation is expressed in factored form $y = a(x - r)(x - s)$, each factor can be used to determine a zero, or *x*-intercept, of the parabola.
- An equation for a parabola can be determined using the zeros and the coordinates of one other point on the parabola.

Need to Know

- If a quadratic relation is expressed in the form $y = a(x - r)(x - s)$,
  - the *x*-intercepts are *r* and *s*
  - the equation of the axis of symmetry is the vertical line defined by the equation $x = (r + s) \div 2$
  - the *x*-coordinate of the vertex is $(r + s) \div 2$
  - the *y*-intercept is $c = a \times r \times s$



CHECK *Your Understanding*

1. Complete the following for each quadratic relation below.

i) Determine the zeros.

ii) Explain how the zeros are related to the factors in the quadratic expression.

iii) Determine the y-intercept.

iv) Determine the equation of the axis of symmetry.

v) Determine the coordinates of the vertex.

vi) Is the graph a parabola? How can you tell?

vii) Sketch the graph.

a) $y = -2x(x + 3)$

b) $y = (x - 3)(x + 1)$

c) $y = 2(x - 1)(x + 2)$

2. Match each quadratic relation with the correct parabola.

a) $y = (x - 2)(x + 3)$

b) $y = (x - 3)(x + 2)$

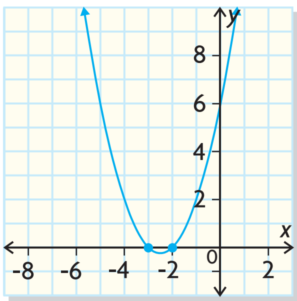
c) $y = (x + 2)(x + 3)$

d) $y = (3 - x)(x + 2)$

e) $y = (3 + x)(2 - x)$

f) $y = (x - 2)(x - 3)$

i)



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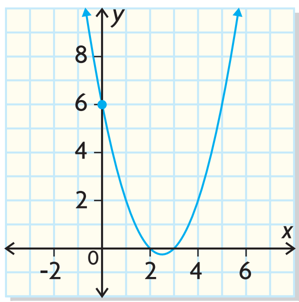

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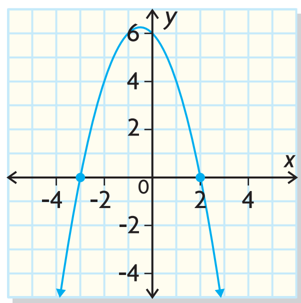

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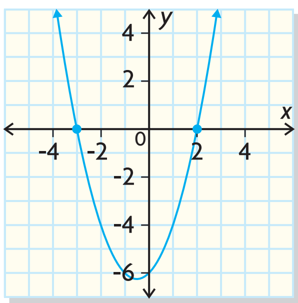

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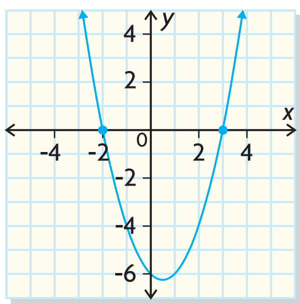

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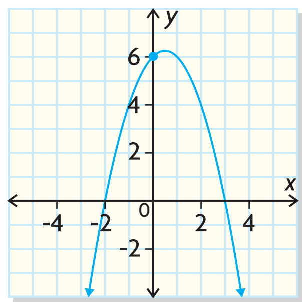


3. A quadratic relation has an equation of the form $y = a(x - r)(x - s)$. The graph of the relation has zeros at (2, 0) and (−6, 0) and passes through the point (3, 5). Determine the value of a.

PRACTISING

4. Determine the y-intercept, zeros, equation of the axis of symmetry, and vertex of each quadratic relation.

a) $y = (x - 3)(x + 3)$

b) $y = (x + 2)(x + 2)$

c) $y = (x - 2)(x - 2)$

d) $y = -(x - 2)(x + 2)$

e) $y = 2(x + 3)^2$

f) $y = -4(x - 4)^2$

5. Sketch the graph of each relation in question 4.

6. A quadratic relation has an equation of the form $y = a(x - r)(x - s)$. Determine the value of a when

a) the parabola has zeros at (4, 0) and (2, 0) and a y-intercept at (0, 1)

b) the parabola has x-intercepts at (4, 0) and (−2, 0) and a y-intercept at (0, −1)

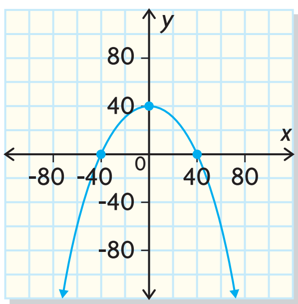
c) the parabola has zeros at (5, 0) and (0, 0) and a minimum value of −10

d) the parabola has x-intercepts at (5, 0) and (−3, 0) and a maximum value of 6

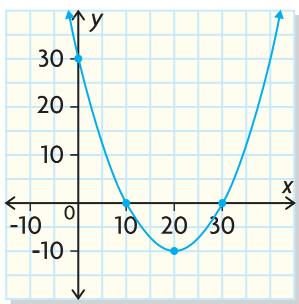
e) the parabola has its vertex at (5, 0) and a y-intercept at (0, −10)

7. **K** Determine the zeros, equation of the axis of symmetry, and vertex of each parabola. Then determine an equation for each quadratic relation.

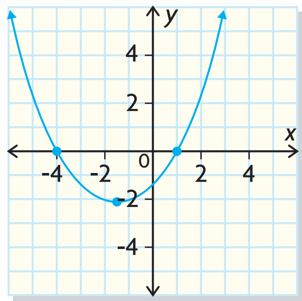
a)



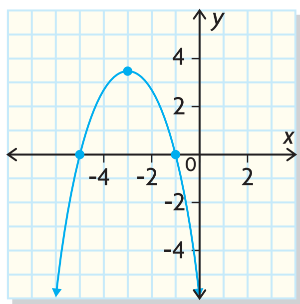
b)



c)



d)



8. **a)** Sketch the graph of $y = a(x - 2)(x + 3)$ when $a = 3$.
b) Describe how your graph for part a) would change if the value of a changed to 2, 1, 0, −1, −2, and −3.

9. **a)** Sketch the graph of $y = (x - 2)(x - s)$ when $s = 3$.
b) Describe how your graph for part a) would change if the value of s changed to 2, 1, 0, 1, −2, and −3.

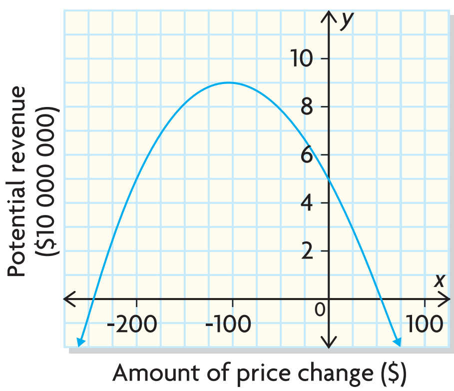
10. The x-intercepts of a parabola are −3 and 5. The parabola crosses the y-axis at −75.
a) Determine an equation for the parabola.
b) Determine the coordinates of the vertex.

11. Sometimes the equation $y = a(x - r)(x - s)$ cannot be used to determine the equation of a parabola from its graph. Explain when this is not possible, and draw graphs to illustrate.

12. A ball is thrown into the air from the roof of a building that is 25 m high. The ball reaches a maximum height of 45 m above the ground after 2 s and hits the ground 5 s after being thrown.

a) Use the fact that the relation between time and the height of the ball is a quadratic relation to sketch an accurate graph of the relation.
b) Carefully fold the graph along its axis of symmetry. Extend the short side of the parabola to match the long side.
c) Where does the extended graph cross the time axis?
d) What are the zeros of the relation?
e) Determine the coordinates of the vertex.
f) Determine an equation for the relation.
g) What is the meaning of each zero?

13. A car manufacturer decides to change the price of its new luxury sedan (LS) model to increase sales. The graph shows the relationship between revenue and the size of the price change.



a) Determine an equation for the graph.
b) How should the price be changed for maximum revenue?

Career Connection

Jobs in the music industry include recording artist, studio musician, songwriter, producer, recording engineer, digital audio workstation operator, music programmer, and re-mixer.

14. **T** Ryan owns a small music store. He currently charges \$10 for each CD. At this price, he sells about 80 CDs a week. Experience has taught him that a \$1 increase in the price of a CD means a drop of about five CDs per week in sales. At what price should Ryan sell his CDs to maximize his revenue?

15. **A** Rahj owns a hardware store. For every increase of 10¢ in the price of a package of batteries, he estimates that sales decrease by 10 packages per day. The store normally sells 700 packages of batteries per day, at \$5.00 per package.

a) Determine an equation for the revenue, y, when x packages of batteries are sold.

b) What is the maximum daily revenue that Rahj can expect from battery sales?

c) How many packages of batteries are sold when the revenue is at a maximum?

16. **C** Create a flow chart that summarizes the process you would use to determine an equation of a parabola from its graph. Assume that the parabola has two zeros.

Extending

17. Without graphing, match each quadratic relation in factored form (column 1) with the equivalent quadratic relation in standard form (column 2). Explain your reasoning.

Column 1	Column 2
a) $y = (2x - 3)(x + 4)$	**i)** $y = 12x^2 - 5x - 3$
b) $y = (3x + 1)(4x - 3)$	**ii)** $y = -2x^2 - 5x + 12$
c) $y = (3 - 2x)(4 + x)$	**iii)** $y = 2x^2 + 11x - 12$
d) $y = (3 - 4x)(1 + 3x)$	**iv)** $y = 2x^2 + 5x - 12$
	v) $y = -12x^2 + 5x + 3$
	vi) $y = 12x^2 + 5x - 3$

18. Martin wants to enclose the backyard of his house on three sides to form a rectangular play area. He is going to use the wall of his house and three sections of fencing. The fencing costs \$15/m, and Martin has budgeted \$720. Determine the dimensions that will produce the largest rectangular area.

3 Mid-Chapter Review

FREQUENTLY ASKED Questions

Q: What are the key properties of a quadratic relation?

A: The key properties are:

- In a table of values, the second differences are constant and not zero.
- The degree of the equation that represents the relation is 2.
- The graph has a U shape, which is called a parabola.
- Every parabola has a vertex that is the highest or lowest point on the curve.
- Every parabola has an axis of symmetry that passes through its vertex.

Study Aid

- See Lesson 3.1 and Lesson 3.2, Examples 1 to 4.
- Try Mid-Chapter Review Questions 1 and 2.

Q: What information can you easily determine from the factored and standard forms of a quadratic relation?

A: From the standard form $y = ax^2 + bx + c$, you can determine the y-intercept, which is c.

From the factored form $y = a(x - r)(x - s)$, you can determine

- the zeros, or x-intercepts, which are r and s
- the equation of the axis of symmetry, which is $x = \frac{r + s}{2}$
- the coordinates of the vertex, by substituting the value of the axis of symmetry for x in the relation
- the y-intercept, which is $a \times r \times s$

From both forms, you can determine the direction in which the parabola opens: upward when $a > 0$ and downward when $a < 0$.

Study Aid

- See Lesson 3.2, Example 2, and Lesson 3.3, Examples 1 to 3.
- Try Mid-Chapter Review Questions 3 to 7.

Q: If you are given information about a quadratic relation, how can you determine the equation?

A: If the graph has zeros, these can be used to write the equation of the quadratic relation in factored form. Then you can use a different point on the parabola to determine the coefficient a.

Study Aid

- See Lesson 3.3, Example 4.
- Try Mid-Chapter Review Questions 8 to 10.

EXAMPLE

The points $(-2, 0)$ and $(3, 0)$ are the zeros of a parabola that passes through $(4, 12)$. Determine an equation for the quadratic relation.

Solution

Use the zeros to write the equation $y = a(x + 2)(x - 3)$. Substitute the coordinates of the point $(4, 12)$ into the equation to determine the coefficient a.

$12 = a(4 + 2)(4 - 3)$

$12 = a(6)(1)$

$2 = a$

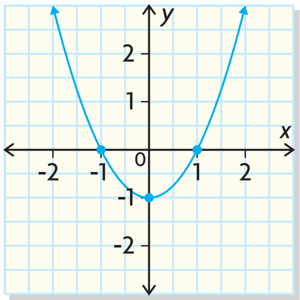
An equation for the quadratic relation is $y = 2(x + 2)(x - 3)$.

PRACTICE Questions

Lesson 3.1

1. State whether each relation is quadratic. Justify your answer.

a)



b) $y = 5x^2 + 3x - 1$

c)

x	0	1	2	3	4	5
y	3	2	1	0	1	2

2. Each table of values represents a quadratic relation. Decide, without graphing, whether the parabola opens upward or downward.

a)

x	−2	−1	0	1	2
y	0	−5	0	15	40

b)

x	−2	−1	0	1	2
y	−3	3	5	3	−3

Lesson 3.2

3. Graph $y = -x^2 + 6x$ to determine

a) the equation of the axis of symmetry

b) the coordinates of the vertex

c) the y-intercept

d) the zeros

4. The points $(-3, 8)$ and $(9, 8)$ lie on opposite sides of a parabola. Determine the equation of the axis of symmetry.

5. Use a graphing calculator to graph each relation. Determine the y-intercept, zeros, equation of the axis of symmetry, and vertex.

a) $y = x^2 + 8x + 15$

b) $y = -2x^2 + 16x - 32$

6. A soccer ball is kicked into the air. Its height, h, in metres, is approximated by the equation $h = -5t^2 + 15t + 0.5$, where t is the time in seconds since the ball was kicked.

a) From what height is the ball kicked?

b) When does the ball hit the ground?

c) When does the ball reach its maximum height?

d) What is the maximum height of the ball?

e) What is the height of the ball at $t = 3$? Is the ball travelling upward or downward at this time? Explain.

f) When is the ball at a height of 10 m?

Lesson 3.3

7. Determine the y-intercept, zeros, equation of the axis of symmetry, and vertex of each quadratic relation. Then sketch its graph.

a) $y = (x - 5)(x + 5)$

b) $y = -(x - 6)(x - 2)$

c) $y = 2(x - 1)(x + 3)$

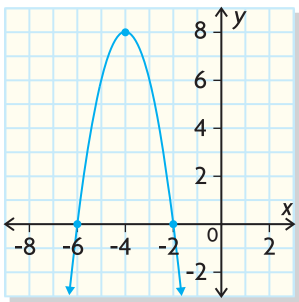
d) $y = -0.5(x + 4)^2$

8. The zeros of a parabola are −10 and 30. The parabola crosses the y-axis at 50.

a) Determine an equation for the parabola.

b) Determine the coordinates of the vertex.

9. Determine an equation for this quadratic relation.



10. Give an example of an equation of a quadratic relation whose vertex and x-intercept occur at the same point.

3.4 Expanding Quadratic Expressions

GOAL

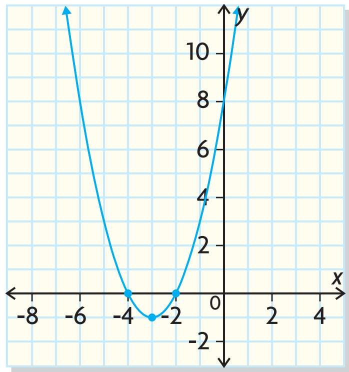
Determine the product of two binomials using a variety of strategies.

YOU WILL NEED

- algebra tiles

LEARN ABOUT the Math

Brandon was doing his math homework. For one question, he had to determine the equation of the parabola shown at the right.



Brandon's answer was $y = (x + 4)(x + 2)$.

His older sister, Devin, said that the answer can also be $y = x^2 + 6x + 8$.

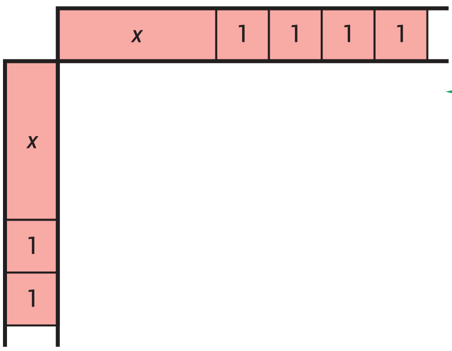
? How can Devin show Brandon that both answers are correct?

EXAMPLE 1 | Connecting an area model to the product of two binomials

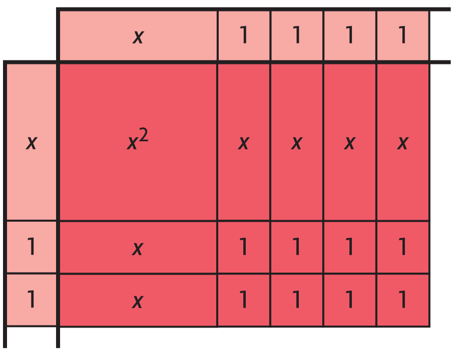
Show that the equations $y = (x + 4)(x + 2)$ and $y = x^2 + 6x + 8$ represent the same quadratic relation.

Devin's Solution

$y = (x + 4)(x + 2)$



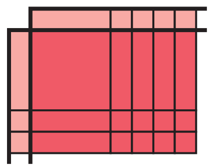
I wanted to show Brandon how to multiply two binomials. I know that the area of a rectangle is the product of its length and its width. I used algebra tiles to represent a width of $x + 2$ and a length of $x + 4$.



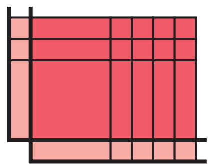
I used x^2 tiles, x tiles, and unit tiles to fill in the area of the rectangle with these dimensions. The area of the rectangle represents the product of the two binomials.

Communication *Tip*

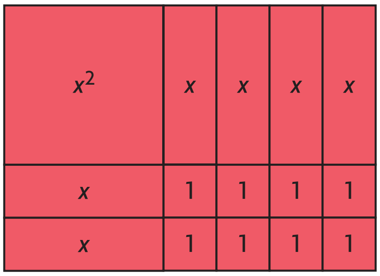
The tiles used for each dimension of a rectangle can be placed on either the left or right, and either on the top or bottom. The resulting area of the rectangle is the same in each case. The only difference occurs in the position of the x^2, x, and unit tiles within the rectangle. For example,



and



represent the same product.



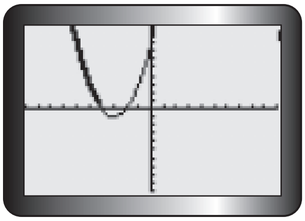
I counted the tiles in the rectangle to get an expression for its area, *A*.

$A = x^2 + 2x + 4x + 8$

$A = x^2 + 6x + 8$

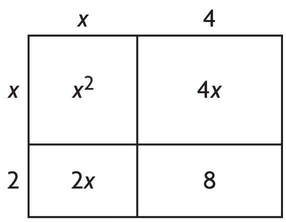
$y = (x + 4)(x + 2)$ and $y = x^2 + 6x + 8$ are the same quadratic relation.

Brandon's equation is in factored form and mine is in standard form.



I graphed both relations to see if they represented the same parabola. The second parabola traced exactly over the first parabola.

Reflecting



A. Why did Devin use only red tiles in her rectangle model?

B. Explain how the area diagram at the left is related to Devin's algebra tile model and the product $(x + 4)(x + 2)$.

C. Is the value of *a* always the same in factored form and standard form if both relations represent the same parabola? Explain.

APPLY the Math

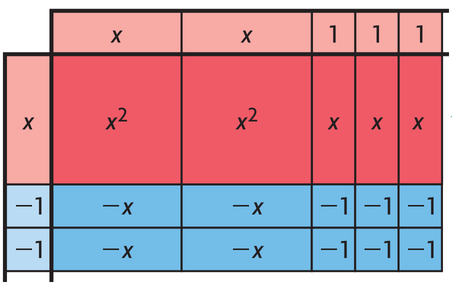
EXAMPLE 2 **Connecting** the product of two binomials to the distributive property

Expand and simplify.

a) $(2x + 3)(x - 2)$ **b)** $(2x - 1)(x - 3)$

Lorna's Solution

a)

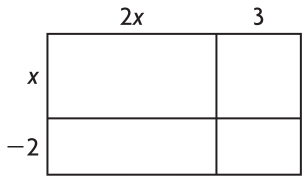


I placed tiles that correspond to the binomial factors along the sides of a rectangle. I represented $x - 2$ as $x + (-2)$ because I didn't know how to remove part of a tile.

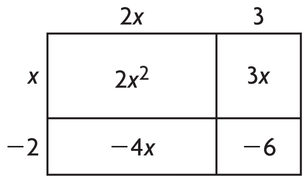
Then I used tiles to fill in the area. The rules for multiplying integers helped me choose the correct colours to use. Since a blue tile is negative and a red tile is positive, I used blue tiles to represent the negative product.

$$(2x + 3)(x - 2) = 2x^2 - 4x + 3x - 6$$
$$= 2x^2 - x - 6$$

I counted the different types of algebra tiles to get the product.



I noticed that the area in the tile model was divided into four sections, so I divided a rectangle into four small rectangles. I labelled the side lengths.



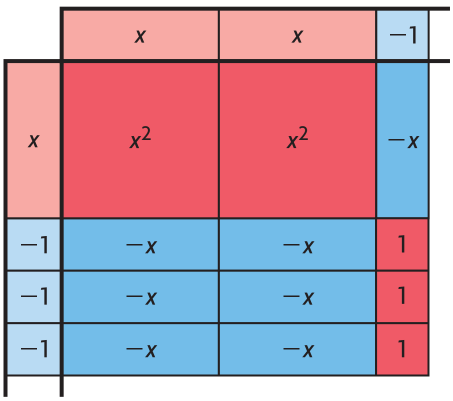
$$(2x + 3)(x - 2) = 2x^2 - 4x + 3x - 6$$
$$= 2x^2 - x - 6$$

I wrote an expression for the area of each small rectangle. The area of the large rectangle is the sum of the areas of the four small rectangles. When I collected like terms, I saw that the product was the same.

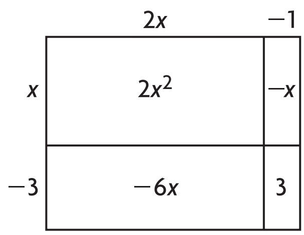
$$(2x + 3)(x - 2) = 2x(x - 2) + 3(x - 2)$$
$$= 2x^2 - 4x + 3x - 6$$
$$= 2x^2 - x - 6$$

I recognized the distributive property in the area model. The areas in the first column show the product $2x(x - 2)$. The areas in the second column show the product $3(x - 2)$. I used the distributive property again. Then I collected like terms to get the final result.

b)



I created an algebra tile model. This time the unit tiles that I used to fill in the area had to be positive red tiles, since the result is the product of two negative blue tiles.



I made an area diagram to show the area of the four sections of the tile model.

$$(2x - 1)(x - 3) = 2x(x - 3) - 1(x - 3)$$
$$= 2x^2 - 6x - x + 3$$
$$= 2x^2 - 7x + 3$$

I could have used the distributive property without a picture or model. I collected like terms to get the final result.

EXAMPLE 3 | **Representing** the product of two binomials symbolically

Multiply each expression.

a) $(x - 5)(x + 5)$ **b)** $(3x - 5)^2$

Zac's Solution

a) $(x - 5)(x + 5) = x^2 + 5x - 5x - 25$
$= x^2 - 25$

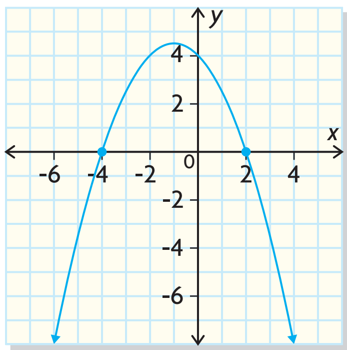
I multiplied each term in the second binomial by x and then by -5. I collected like terms and got a binomial for my final result.

b) $(3x - 5)^2 = (3x - 5)(3x - 5)$
$= 9x^2 - 15x - 15x + 25$
$= 9x^2 - 30x + 25$

I wrote the expression as a product of two binomials. I multiplied each term in the second binomial by $3x$ and then by -5. I collected like terms and got a trinomial for my final result.

EXAMPLE 4 | **Connecting** the factored form and standard form of a quadratic relation

Determine the equation of the parabola. Express your answer in standard form.



Mathieu's Solution

$y = a[x - (-4)](x - 2)$
$y = a(x + 4)(x - 2)$

I wrote the equation in factored form using the zeros of the parabola. Then I wrote an equivalent expression for $x - (-4)$.

$x = 0, y = 4$
$4 = a(0 + 4)(0 - 2)$
$4 = a(4)(-2)$
$4 = -8a$
$\frac{4}{-8} = \frac{-8a}{-8}$
$-0.5 = a$

There is only one value of a that gives a parabola with these zeros and y-intercept. To determine this value, I substituted the coordinates of the y-intercept (0, 4) into the equation and solved for a.

$$y = -0.5(x + 4)(x - 2)$$
$$y = -0.5(x^2 - 2x + 4x - 8)$$
$$y = -0.5x^2 + x - 2x + 4$$
$$y = -0.5x^2 - x + 4$$

I substituted the value of *a* into the factored form of the equation. I multiplied the two binomials. Then I multiplied all the terms by −0.5 and collected like terms to get the result in standard form.

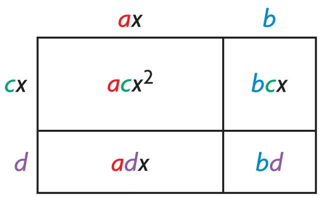
In Summary

Key Ideas

- Quadratic expressions can be expanded using the distributive property, then simplified by collecting like terms.
- An area diagram or algebra tiles can be used to show the relation between two binomial factors of degree one and their product.

Need to Know

- To calculate the product of two binomials, use the distributive property twice.

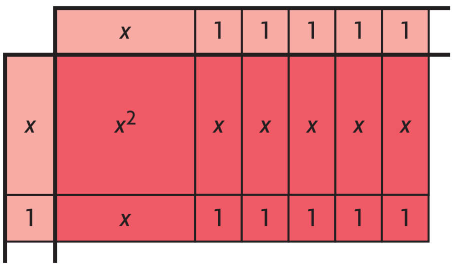


$$(ax + b)(cx + d) = ax(cx + d) + b(cx + d)$$
$$= acx^2 + adx + bcx + bd$$

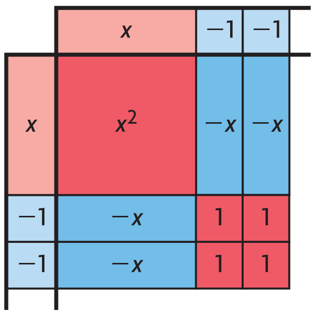
CHECK *Your Understanding*

1. State the binomials that are represented by the length and width of each rectangle. Then determine the product that is represented by the area.

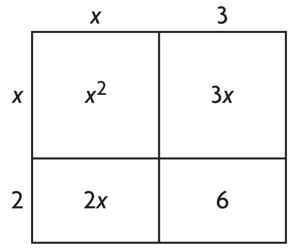
a)



b)



2. Copy and complete this table.

	Expression	Area Diagram	Expanded and Simplified Form
	$(x + 2)(x + 3)$		$x^2 + 5x + 6$
a)	$(x + 1)(x + 6)$		
b)	$(x + 1)(x - 4)$		
c)	$(x - 2)(x + 2)$		
d)	$(x - 3)(x - 4)$		
e)	$(x + 2)(x + 4)$		
f)	$(x - 2)(x - 6)$		

PRACTISING

3. Determine the missing terms.

a) $(m + 3)(m + 2) = ■ + 2m + 3m + ●$

b) $(k - 2)(k + 1) = ■ + ● - 2k - 2$

c) $(r + 4)(r - 3) = r^2 - 3r + ■ - ●$

d) $(x - 5)(x - 2) = x^2 - ■ - ● + 10$

e) $(2n + 1)(3n - 2) = ■ - ● + 3n - 2$

f) $(5m - 2)(m - 3) = 5m^2 - ■ - 2m + ●$

4. Expand and simplify.

a) $(x + 2)(x + 5)$ **c)** $(x + 2)(x - 3)$ **e)** $(x - 4)(x - 2)$

b) $(x + 2)(x + 1)$ **d)** $(x + 2)(x - 1)$ **f)** $(x - 5)(x - 3)$

5. Expand and simplify.

a) $(5x + 2)(x + 2)$ **c)** $(x - 2)(7x + 3)$ **e)** $(x - 2)(4x - 6)$

b) $(x + 2)(4x + 1)$ **d)** $(3x - 2)(x + 1)$ **f)** $(7x - 5)(x - 3)$

6. Expand and simplify.

a) $(x + 3)(x - 3)$ **c)** $(2x - 1)(2x + 1)$ **e)** $(4x - 6)(4x + 6)$

b) $(x + 6)(x - 6)$ **d)** $(3x - 3)(3x + 3)$ **f)** $(7x - 5)(7x + 5)$

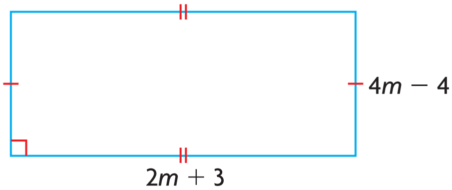
7. Expand and simplify.

a) $(x + 1)^2$ **c)** $(c - 1)^2$ **e)** $(6z - 5)^2$

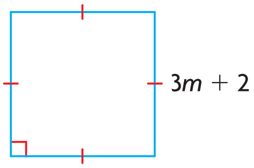
b) $(a + 4)^2$ **d)** $(5y - 2)^2$ **f)** $(-3d + 5)^2$

8. Write a simplified expression for the area of each figure.

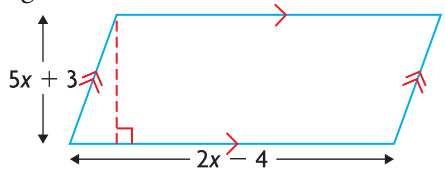
A **a)**



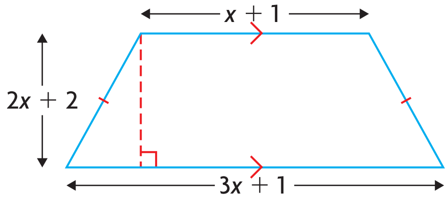
b)



c)



d)



9. Expand and simplify.

a) $4(x - 6)(x + 7)$

b) $-(x + 3)(4x - 1)$

c) $6x(x + 1)^2$

d) $(x + 4)(x - 2) + (x - 1)(x + 5)$

e) $(4x - 1)(4x + 1) - (x + 3)^2$

f) $2(3x + 4)^2 - 3(x - 2)^2$

10. Expand and simplify.

a) $(x + y)(2x + 3y)$

b) $(x + 2y)(3x + y)$

c) $(3x - 2y)(5x + 4y)$

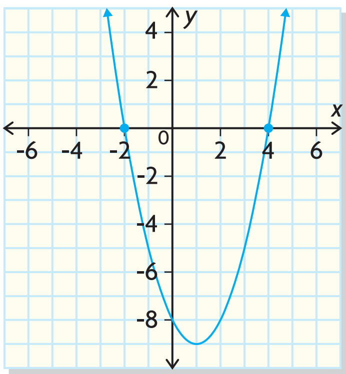
d) $(8x - y)(7x + 2y)$

e) $(6x - 5y)(6x + 5y)$

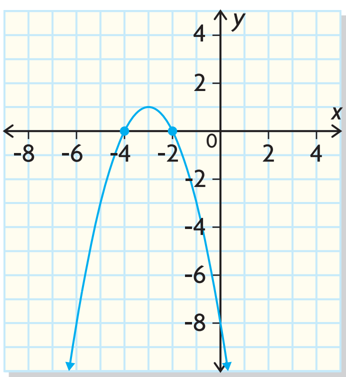
f) $(9x - 7y)^2$

11. Determine the equation of each parabola. Express the equation
K in standard form.

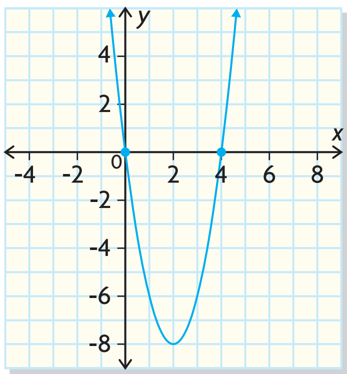
a)



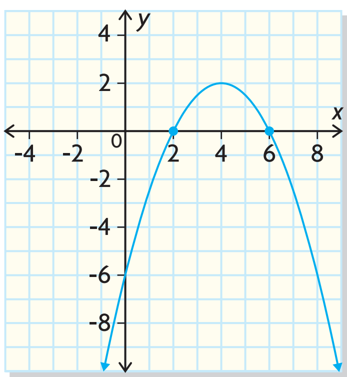
b)



c)



d)



12. Write each quadratic relation in standard form. State which way the parabola opens.

	Zeros	A Point on the Graph
a)	−1 and 7	(3, 5)
b)	−1 and −5	(−3, −4)
c)	3 and 7	(0, 3)
d)	−2 and 6	(−1, −1)
e)	−2 and 8	(3, 7)

13. The area of a rectangle is represented by the expression $2x^2 + 14x + 20$. Bill claims that this rectangle could have either the dimensions $(2x + 4)$ and $(x + 5)$ or the dimensions $(2x + 10)$ and $(x + 2)$. Do you agree or disagree? Justify your opinion.

14. **C** Explain how you know that the product will be quadratic when you expand $(12x - 7)(5x + 1)$.

15. **T** The Rainbow Bridge in Utah, shown at the left, is a natural arch that is approximately parabolic in shape. The arch is about 88 m high. It is 84 m across at its base. Determine a quadratic relation, in standard form, that models the shape of the arch.

16. Jay claims that whenever two binomials are multiplied together, the result is always a trinomial. Is his claim correct? Use examples to support your decision.

Extending

17. Expand and simplify each expression.
a) $(x + 3)^3$
b) $(2x - 2)^3$
c) $(4x + 2y)^3$
d) $[(x + 2)(x - 2)]^2$
e) $(x + 6)(x + 3)(x - 6)(x - 3)$
f) $(3x^2 + 6x - 1)^2$

18. Expand each expression.
a) $(a + b)^1$
b) $(a + b)^2$
c) $(a + b)^3$
d) $(a + b)^4$

19. Discuss any patterns you see in question 18.

3.5 Quadratic Models Using Factored Form

GOAL

Determine the equation of a quadratic model using the factored form of a quadratic relation.

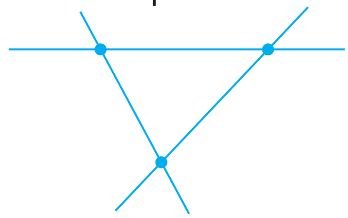
YOU WILL NEED

- graphing calculator
- grid paper
- ruler

INVESTIGATE the Math

You can draw one straight line through any pair of points. If you have three points you can draw a maximum of three lines. The maximum number of lines possible occurs when the points do not lie on the same line.

3 points



? What is the maximum number of lines you can draw using 100 points?

A. Can you answer the question directly using a diagram? Explain.

B. Since two points are needed to draw a line, using zero and one point results in zero lines. Copy and complete the rest of the table by drawing each number of points and determining the maximum number of lines that can be drawn through pairs of points.

Number of Points, *x*	0	1	2	3	4	5	6
Maximum Number of Lines, *y*	0	0					

C. Use your data to create a scatter plot with an appropriate scale.

D. What shape best describes your graph? Draw a **curve of good fit**.

curve of good fit
a curve that approximates, or is close to, the distribution of points in a scatter plot

E. Carry out appropriate calculations to determine whether the curve you drew for part D is approximately linear, approximately quadratic, or some other type.

F. What are the zeros of your curve? Use the zeros to write an equation for the relation in factored form: $y = a(x - r)(x - s)$.

G. Use one of the ordered pairs in your table (excluding the zeros) to calculate the value of *a*. Write an equation for the relation in both factored form and standard form.

H. Use a graphing calculator and **quadratic regression** to determine the equation of this quadratic relation model.

quadratic regression
a process that fits the second degree relation $y = ax^2 + bx + c$ to the data

curve of best fit

the curve that best describes the distribution of points in a scatter plot, usually found using a process called regression

Tech Support

For help using a TI-83/84 graphing calculator to determine the equation of a curve of best fit using quadratic regression, see Appendix B-10. If you are using a TI-*n*spire, see Appendix B-46.

I. How does your equation compare with the graphing calculator's **curve of best fit** equation?

J. Use your equation to predict the number of lines that can be drawn using 100 points.

Reflecting

K. How does the factored form of a quadratic relation help you determine the equation of a curve of good fit when it has two zeros?

L. How would the equation change if the data were quadratic and the curve of good fit had only one zero?

M. If a curve of good fit for a set of data had no zeros, could the factored form be used to determine its equation? Explain.

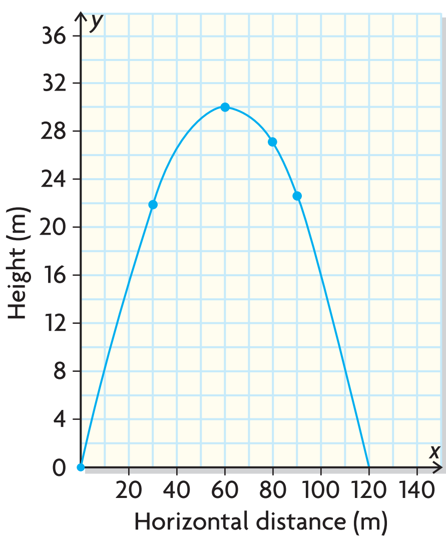
APPLY the Math

EXAMPLE 1 **Connecting** the zeros and factored form to an equation that models data

Data from the flight of a golf ball are given in this table. If the maximum height of the ball is 30.0 m, determine an equation for a curve of good fit.

Horizontal Distance (m)	0	30	60	80	90
Height (m)	0.0	22.0	30.0	27.0	22.5

Jill's Solution



I plotted the data. I drew a parabola as a curve of good fit because it seemed to be close to most of the data. The maximum height was 30.0 m, so the vertex of the parabola is located at (60, 30). The equation of the axis of symmetry is $x = 60$.

A zero occurs at (0, 0). Since a parabola is symmetric, I determined that another zero is located at (120, 0).

$$y = a(x - 0)(x - 120)$$

I wrote a general equation of the parabola in factored form. Because the zeros are 0 and 120, I knew that $(x - 0)$ and $(x - 120)$ are factors.

$$30 = a(60)(60 - 120)$$
$$\frac{30}{60(-60)} = a$$
$$\frac{1}{2(-60)} = a$$
$$-\frac{1}{120} = a$$

I substituted (60, 30) into the equation, since it is a point on the curve. Then I solved for *a*.

$$y = -\frac{1}{120}x(x - 120)$$
$$y = -\frac{1}{120}x^2 + x$$

I used the value of *a* to write the equation. Then I expanded the equation to write it in standard form.

When $x = 30$,

$$y = -\left(\frac{1}{120}\right)(30)(30 - 120)$$
$$y = -\left(\frac{1}{4}\right)(-90)$$
$$y = 22\frac{1}{2}$$

I checked the equation by substituting other values of *x* into it.

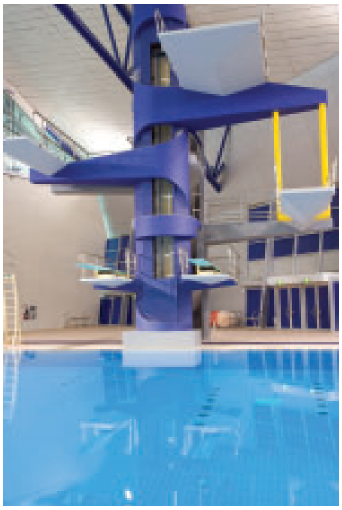
When $x = 80$,

$$y = -\left(\frac{1}{120}\right)(80)(80 - 120)$$
$$y = -\left(\frac{2}{3}\right)(-40)$$
$$y = 26\frac{2}{3}$$

The points $\left(30, 22\frac{1}{2}\right)$ and $\left(80, 26\frac{2}{3}\right)$ from the equation are close to the points (30, 22.0) and (80, 27.0) from the data.

The results for *y* were close to the values in the table, so the equation for the curve of good fit is reasonable.

An equation of good fit is $y = -\frac{1}{120}x^2 + x$.



EXAMPLE 2 Selecting an informal strategy to determine an equation of a curve of good fit

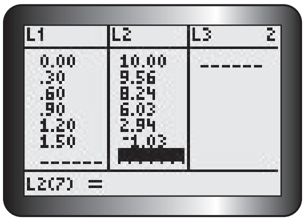
A competitive diver does a handstand dive from a 10 m platform. This table of values shows the time in seconds and the height of the diver, relative to the surface of the water, in metres.

Time (s)	0	0.3	0.6	0.9	1.2	1.5
Height (m)	10.00	9.56	8.24	6.03	2.94	−1.03

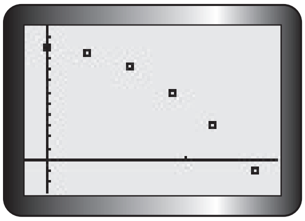
Determine an equation that models the height of the diver above the surface of the water during the dive. Verify your result using quadratic regression.

Madison's Solution

> **Tech Support**
> For help using a TI-83/84 graphing calculator to create a scatter plot, see Appendix B-10. If you are using a TI-*n*spire, see Appendix B-46.



I entered the data in the lists of a graphing calculator and created a scatter plot.



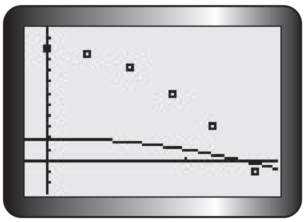
The points looked like they formed half of a parabola. I assumed that the diver was at the maximum height at the start of the dive. This meant the vertex was located at (0, 10). I estimated that one zero occurred at 1.4 and the other zero occurred at −1.4 since the *y*-axis is the axis of symmetry.

$$y = a(x - 1.4)(x + 1.4)$$

I wrote an equation of the parabola in factored form.

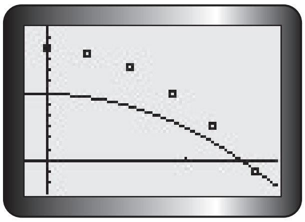
$$y = -1(x - 1.4)(x + 1.4)$$

I entered my equation into the equation editor using −1 as a guess for the value of *a*. I knew that *a* is negative since the parabola opens downward.



This graph is not a good fit.

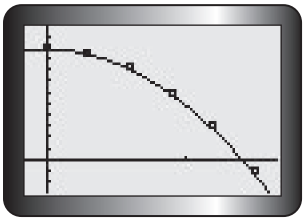
$y = -3(x - 1.4)(x + 1.4)$



I tried -3 as a value of *a* and graphed the relation again.

This graph is not a good fit either.

$y = -5(x - 1.4)(x + 1.4)$

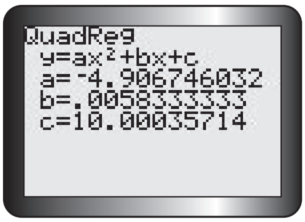


I tried -5 as a value of *a* and graphed the relation again.

This graph is a good fit that models the height of the diver above the surface of the water during the dive.

$$y = -5(x^2 + 1.4x - 1.4x - 1.96)$$
$$y = -5x^2 + 9.8$$

I expanded my equation to write it in standard form.



Then I used quadratic regression to determine the equation of the curve of best fit. My equation and the calculator's equation are very close.

EXAMPLE 3 Solving a problem using a model

Jeff and Tim are analyzing data collected from a motion detector following the launch of their model rocket.

Time (s)	0.0	1.0	2.0	3.0	4.0
Height (m)	0.0	16.0	20.0	15.5	0.0

a) Determine an equation for a curve of good fit.

b) Use the equation you determined for part a) to estimate the height of the rocket 0.5 s after it is launched.

Phil's Solution

a) A parabola might model this situation.

Since the height of the rocket increased and then decreased, I assumed that a quadratic model might be reasonable.

$$y = a(x - 0)(x - 4)$$
$$y = ax(x - 4)$$

I wrote a general equation of the relation in factored form. The zeros are 0 and 4, so $(x - 0)$ and $(x - 4)$ are factors.

$$20 = a(2)(2 - 4)$$
$$\frac{20}{(2)(-2)} = a$$
$$-5 = a$$

To determine the value of a, I substituted (2, 20) into the equation since it is a point on the curve.

$$y = -5x(x - 4)$$

I used the value of a to write the final equation of the curve of good fit.

When $x = 2$,
$y = -5(2)(2 - 4)$
$y = -5(2)(-2)$
$y = 20$

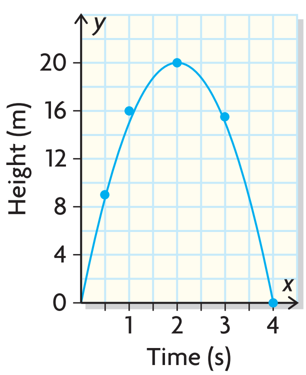
When $x = 3$,
$y = -5(3)(3 - 4)$
$y = -5(3)(-1)$
$y = 15$

I checked my equation by substituting other values of x into it. The results for y were close to the values in the table so my equation seems reasonable.

b) When $x = 0.5$,
$y = -5(0.5)(0.5 - 4)$
$y = -5(0.5)(-3.5)$
$y = 8.75$

I substituted 0.5 for x into the equation for the curve of good fit.

The height of the rocket after 0.5 s is approximately 8.8 m.



I checked the result by plotting the points and drawing the graph. The fit seems reasonable.

In Summary

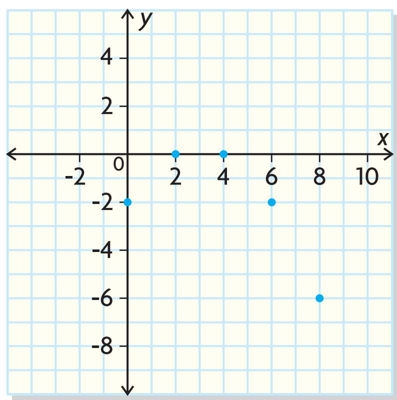
Key Idea

- If a curve of good fit for data with a parabolic pattern passes through the horizontal axis, then the factored form of the quadratic relation can be used to determine an algebraic model for the relationship.

Need to Know

- The estimated or actual x-intercepts, or zeros, of a curve of good fit represent the values of r and s in the factored form of the quadratic relation $y = a(x - r)(x - s)$.
- The value of a can be determined algebraically by substituting the coordinates of a point (other than a zero) that lies on or close to the curve of good fit into the equation and then solving for a.
- The value of a can be determined graphically by estimating the value of a and graphing the resulting parabola with graphing technology. By observing the graph, you can adjust your estimate of a and graph again until the parabola passes through or close to a large number of points in the scatter plot.
- Graphing technology can be used to determine an algebraic model for the curve of best fit. You can use quadratic regression when the data has a parabolic pattern.

CHECK Your Understanding



1. a) Use the graph at the right to determine an equation for a curve of good fit. Write the equation in factored and standard forms.

b) Use your equation to estimate the value of y when $x = 1$.

2. a) These data represent the path of a soccer ball as it flies through the air. Create a scatter plot, and then determine an equation for a quadratic curve of good fit.

Horizontal Distance (m)	0.0	1.0	2.0	3.0	4.0
Height (m)	1.0	1.6	1.9	1.6	1.0

b) Use your equation for part a) to estimate the height of the ball when its horizontal distance is 1.5 m.

3. a) Determine the equation of the quadratic curve of best fit for the data.

x	−1	0	1	2	3
y	−2.4	−3.6	−3.6	−2.3	0.1

b) Use your equation for part a) to estimate the value of y when $x = 3.2$.

PRACTISING

4. A parabola passes through the points $(-4, 10)$, $(-3, 0)$, $(-2, -6)$, $(-1, -8)$, $(0, -6)$, $(1, 0)$, and $(2, 10)$.
K

a) Determine an equation for the parabola in factored form.

b) Express your equation in standard form.

c) Use a graphing calculator and quadratic regression to verify the accuracy of the equation you determined.

5. A water balloon was launched from a catapult. The table shows the data collected during the flight of the balloon using stop-motion photography.

Horizontal Distance (m)	0	6	12	18	24	30	36	42	48	54
Height (m)	0.0	11.6	20.4	26.4	29.5	29.7	27.1	21.6	13.3	2.1

a) Use the data to create a scatter plot. Then draw a curve of good fit.

b) Determine an equation for the curve you drew.

c) Estimate the horizontal distance of the balloon when it reached its maximum height. Then use your equation to calculate its maximum height.

d) Use your equation to determine the height of the balloon when its horizontal distance was 40 m.

6. An emergency flare was shot into the air from the top of a building. The table gives the height of the flare at different times during its flight.
A

Time (s)	0	1	2	3	4	5	6
Height (m)	60	75	80	75	60	35	0

a) How tall is the building?

b) Use the data in the table to create a scatter plot. Then draw a curve of good fit.

c) Determine an equation for the curve you drew.

d) Use your equation to determine the height of the flare at 2.5 s.

7. A hang-glider was launched from a platform on the top of the Niagara Escarpment. The data describe the first 13 s of the flight. The values for height are negative whenever the hang-glider was below the top of the escarpment.

Time (s)	0	1	2	3	4	5	6	7	8	9	10	11	12	13
Height (m)	10.0	−0.8	−9.2	−15.2	−18.8	−20.0	−18.8	−15.2	−9.2	−0.8	10.0	23.2	38.8	56.8

a) Determine the height of the platform.
b) Determine an equation that models the height of the hang-glider over the 13 s period.
c) Determine the lowest height of the hang-glider and when it occurred.

8. The data in the table at the right represent the height of a golf ball at different times.

Time (s)	Height (m)
0.0	0.000
0.5	10.175
1.0	17.900
1.5	23.175
2.0	26.000
2.5	26.375
3.0	24.300
3.5	19.775
4.0	12.800
4.5	3.375

a) Create a scatter plot, and draw a curve of good fit.
b) Use your graph for part a) to approximate the zeros of the relation.
c) Determine an equation that models this situation.
d) Use your equation for part c) to estimate the maximum height of the ball.

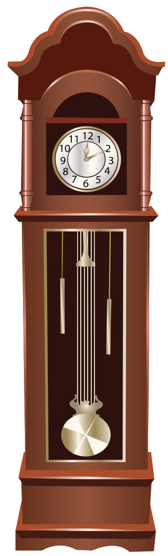
9. For a school experiment, Nichola recorded the height of a model rocket during its flight. The motion detector stopped working, however, during her experiment. The following data were collected before the malfunction.

Time (s)	0.0	1.0	2.0	3.0	4.0
Height (m)	2.00	19.5	27.0	24.5	12.0

a) The height–time relation is quadratic. Determine an equation for the height–time relation.
b) Use the equation you determined for part a) to estimate the height of the rocket at 3.8 s.
c) Determine the maximum height of the rocket. When did the rocket reach its maximum height?

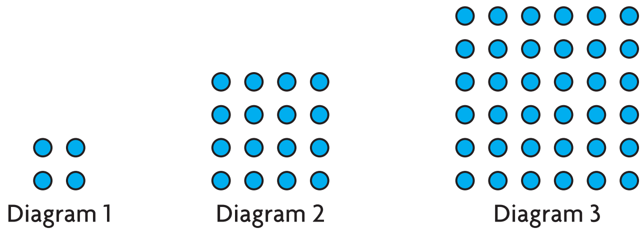
10. A pendulum swings back and forth. The time taken to complete one back-and-forth swing is called the period.

Period (s)	0.5	1.0	1.5	2.0	2.5
Length of Pendulum (cm)	6.2	24.8	55.8	99.2	155.0



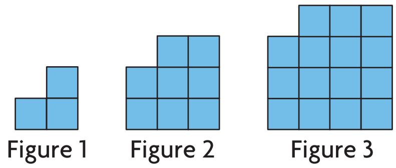
a) Can the data be represented by a quadratic relation? How do you know?
b) Use the data to draw a scatter plot. Then sketch a curve of good fit.
c) Assuming that your graph is a parabola with vertex (0, 0), determine an equation for your curve of good fit.
d) Estimate the period for a pendulum that is 80.0 cm long.
e) Estimate the length of a pendulum that has a period of 2.3 s.

11. Examine this square dot pattern.
T



How many dots are in the 20th diagram? Justify your answer.

12. Examine these three figures made of squares.



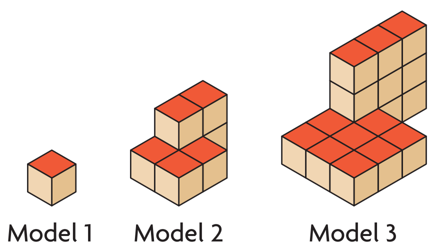
a) Create a table of values to compare the figure number, x, with the area, y. Draw figures 4 and 5 and add this data to your table.
b) Create a difference table to show that the relationship between the figure number and the area is quadratic.
c) Determine an equation for this relationship.
d) Using your difference table, work backwards to determine the zeros of this relationship.
e) Verify that the zeros you determined correspond to your equation.
f) What restriction must be placed on x to model this relationship accurately?

13. Can the factored form of a quadratic relation always be used to model
C a curve of good fit for data that appear to be quadratic? Explain.

14. Create a flow chart that summarizes the steps for determining the equation of a parabola of good fit using the factored form of a quadratic relation.

Extending

15. Examine this pattern of cube structures.



a) Determine the number of cubes in the 15th model.
b) Which model in this pattern could you build using 1249 cubes?

3.6 Exploring Quadratic and Exponential Graphs

GOAL

Compare the graphs of $y = x^2$ and $y = 2^x$ to determine the meanings of zero and negative exponents.

YOU WILL NEED

- graphing calculator

EXPLORE the Math

When you fold a piece of paper in half, you create two regions of equal area. The number of regions increases each time you make a new fold.

? What is the relation between the number of regions and the number of folds, and how does it compare to $y = x^2$?

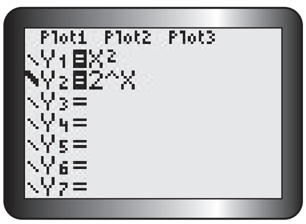
A. Copy and complete the table. Discuss any patterns you see.

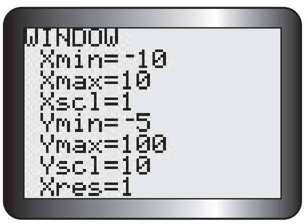
Number of Folds	1	2	3	4	5	6	7
Number of Regions	2						

B. Is the relationship between the number of regions and the number of folds quadratic? Explain.

C. If x represents the number of folds and y represents the number of regions, show that the equation $y = 2^x$ fits the data you found.

D. On a graphing calculator, enter the equation $y = x^2$ into Y1 of the equation editor. Then enter $y = 2^x$ into Y2. Change the line to a thick line for Y2. Use the window settings shown to graph both relations.





E. Discuss how the graphs are the same and how they are different.

Tech *Support*

For help graphing relations and changing window settings using a TI-83/84 graphing calculator, see Appendix B-2 and B-4. If you are using a TI-*n*spire, see Appendix B-38 and B-40.

Tech | ***Support***

For help with the Table feature using a TI-83/84 graphing calculator, see Appendix B-6. If you are using a TI-*n*spire, see Appendix B-42.

F. Create a table of values using the Table feature. Use a starting value of -5 and an increment of 1. Scroll down the X column in your table to compare the y-values of the two relations. Which relation grows faster as x becomes greater?

G. Scroll down the X column in your table. Find the corresponding number in Y2 to determine the value of the power.

i) 0, to determine the value of 2^0

ii) -1, to determine the value of 2^{-1}

iii) -2, to determine the value of 2^{-2}

iv) -3, to determine the value of 2^{-3}

Tech | ***Support***

To change a decimal to a fraction on a TI-83/84 graphing calculator, return to the home screen by pressing [2ND] [MODE]. Then enter the value in the home screen and press [MATH] [ENTER] [ENTER].

H. Express each decimal for part G as a fraction. Rewrite each fraction by changing the denominator to a power of 2.

I. Based on your answers for parts G and H, make conjectures about these values.

i) 3^0 and 5^0

ii) 3^{-1} and 5^{-1}

iii) 3^{-2} and 5^{-2}

iv) 3^{-3} and 5^{-3}

J. Summarize the differences between $y = 3^x$ and $y = 5^x$ by using their graphs to determine

i) symmetry

ii) any x- and y-intercepts

iii) when the y-values are increasing

iv) when the y-values are decreasing

v) what happens to the y-values as x gets larger in the positive direction

vi) what happens to the y-values as x gets larger in the negative direction

Reflecting

K. Will the graph of $y = 2^x$ ever touch the x-axis? Explain.

L. If a is any non-zero base, explain how to write each of the following in rational form.

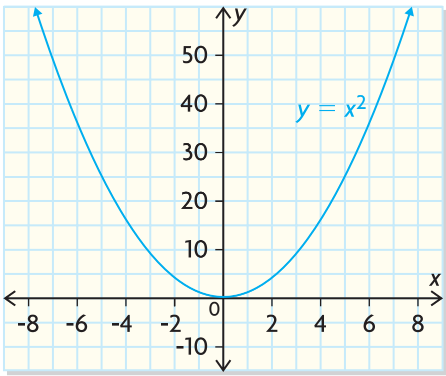
i) a^0 **ii)** a^{-1} **iii)** a^{-2} **iv)** a^{-n}

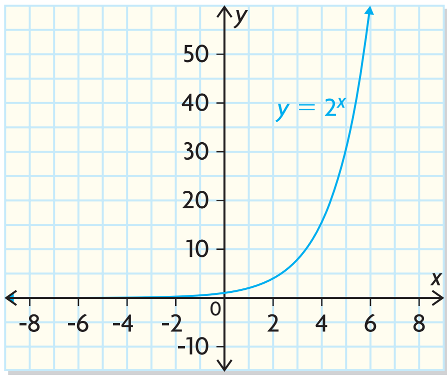
M. In the relations $y = x^2$ and $y = 2^x$, are the y-values ever negative? Explain.

In Summary

Key Ideas

- Patterns in the table of values for $y = 2^x$ can be used to determine the meanings of a^{-n} and a^0 for $a \neq 0$.
- The relations $y = x^2$ and $y = 2^x$ have the following characteristics:





- The graph is symmetric about the y-axis.
- The graph has an x-intercept of 0 and a y-intercept of 0.
- The y-values decrease and then increase as x increases.
- As x increases in the positive direction, the y-values increase.
- As x increases in the negative direction, the y-values increase.

- The graph is not symmetric.
- The graph has no x-intercept and a y-intercept of 1.
- The y-values increase as x increases.
- As x increases in the positive direction, the y-values increase much faster than the y-values for $y = x^2$.
- As x increases in the negative direction, the y-values decrease toward 0.

Need to Know

- When a non-zero base is raised to the exponent 0, the result is 1: $a^0 = 1$ for $a \neq 0$.
- When a non-zero base is raised to a negative exponent, the result is the reciprocal of the base raised to the opposite exponent: $a^{-n} = \frac{1}{a^n}$ for $a \neq 0$.

FURTHER Your Understanding

1. For part I, you made a conjecture about the value of powers with the exponent 0. You can confirm your conjecture using exponent rules.

a) Express $\frac{3^4}{3^4}$ as a single power using the division rule for exponents.

b) Rewrite the numerator and denominator of the expression $\frac{3^4}{3^4}$ in factored form. Simplify where possible. What is the value of this expression?

c) Based on your results for parts a) and b), what can you conclude?

d) Repeat parts a) to c) using the expression $\frac{5^3}{5^3}$.

2. For part I, you made conjectures about the values of powers with negative exponents. These conjectures can also be confirmed using exponent rules.

a) Express $\frac{3^3}{3^4}$ as a single power using the division rule for exponents.

b) Rewrite the numerator and denominator of the expression $\frac{3^3}{3^4}$ in factored form. Simplify where possible. What is the value of this expression?

c) Based on your results for parts a) and b), what can you conclude?

d) Repeat parts a) to c) using the expression $\frac{5^2}{5^4}$.

e) Repeat parts a) to c) using the expression $\frac{5^2}{5^5}$.

3. Evaluate each power. Express your answer in rational form.

a) 2^{-4} **c)** 8^0 **e)** 3^{-4}
b) 4^{-1} **d)** 5^{-2} **f)** 7^{-2}

4. Evaluate each power. Express your answer in rational form.

a) $(-2)^{-5}$ **c)** -7^0 **e)** $(-3)^{-2}$
b) -4^{-2} **d)** -5^{-1} **f)** $(-4)^{-3}$

5. Evaluate each power. Express your answer in rational form.

a) $\left(\frac{1}{2}\right)^2$ **b)** $\left(\frac{1}{2}\right)^{-2}$ **c)** $\left(\frac{2}{3}\right)^3$ **d)** $\left(\frac{2}{3}\right)^{-3}$ **e)** $-\left(\frac{3}{4}\right)^{-2}$ **f)** $\left(-\frac{3}{4}\right)^{-2}$

6. Determine the value of n that makes each statement true.

a) $2^n = \frac{1}{8}$ **c)** $5^n = 1$ **e)** $-3^n = -\frac{1}{9}$
b) $4^n = 64$ **d)** $n^{-3} = \frac{1}{27}$ **f)** $(-n)^4 = 16$

7. Which do you think is greater: 5^{-2} or 10^{-2}? Justify your decision.

8. Which do you think is less: $(-1)^{-100}$ or $(-1)^{-101}$? Justify your decision.

3 Chapter Review

FREQUENTLY ASKED Questions

Q: How do you determine the product of two binomials?

A: You can use algebra tiles or an area diagram, or you can multiply symbolically. All three strategies involve the distributive property.

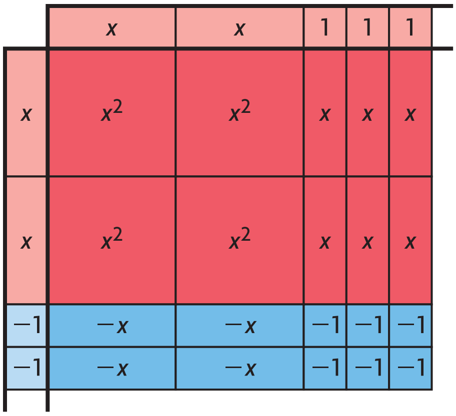
> **Study Aid**
> - See Lesson 3.4, Examples 1 to 3.
> - Try Chapter Review Questions 13 to 15.

EXAMPLE

Expand and simplify $(2x + 3)(2x - 2)$.

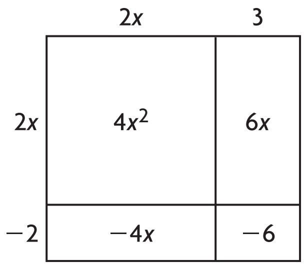
Solution

Using Algebra Tiles



$= 4x^2 - 4x + 6x - 6$
$= 4x^2 + 2x - 6$

Using an Area Diagram



$= 4x^2 - 4x + 6x - 6$
$= 4x^2 + 2x - 6$

Multiplying Symbolically

$(2x + 3)(2x - 2)$

$= 2x(2x - 2) + 3(2x - 2)$
$= 4x^2 - 4x + 6x - 6$
$= 4x^2 + 2x - 6$

Q: How can you determine whether a quadratic model can be used to represent data?

A1: Use the data to create a scatter plot, and draw a curve of good fit. Confirm that your curve of good fit is a parabola.

A2: Create a difference table to see if the second differences are approximately constant.

Q: How can you determine the equation of a parabola of good fit in standard form?

A: Use the data to create a scatter plot. Estimate the zeros of the parabola, and then write a general equation in factored form: $y = a(x - r)(x - s)$. Then substitute the coordinates of a point that is on, or very close to, the curve of good fit. Substitute the value you calculated for a into your equation. Expand and simplify your equation to write it in standard form: $y = ax^2 + bx + c$.

> **Study Aid**
> - See Lesson 3.4, Example 4, and Lesson 3.5, Examples 1 to 3.
> - Try Chapter Review Questions 16 to 18.

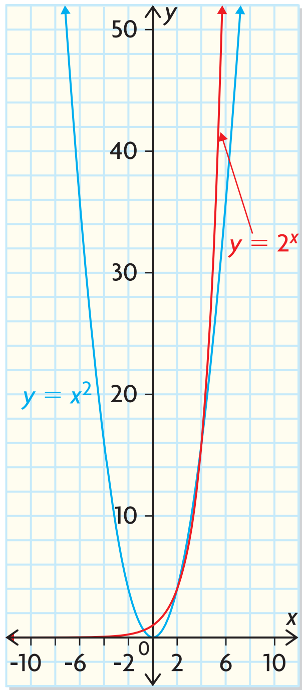
You can check the accuracy of your equation by comparing it with the equation determined using graphing technology and quadratic regression. (Note: This only works when the zeros of the curve of good fit can be estimated or determined.)

Study | Aid

- See Lesson 3.6.

Q: How are $y = x^2$ and $y = 2^x$ different?

A: You can see the differences in their graphs.



- The graph of $y = x^2$ is a parabola with y-values that decrease and then increase as you move from left to right along the x-axis. The graph of $y = 2^x$ is a curve with y-values that always increase as you move from left to right along the x-axis.
- The graph of $y = x^2$ has a vertex and a minimum value of 0. The graph of $y = 2^x$ has no vertex. It approaches a minimum value of 0 but will never equal 0.
- The graph of $y = x^2$ has an x-intercept of 0 and a y-intercept of 0. The graph of $y = 2^x$ has no x-intercept and a y-intercept of 1.
- As x increases in the positive direction, the y-values for $y = 2^x$ increase much faster than the y-values for $y = x^2$.

Study | Aid

- See Lesson 3.6.
- Try Chapter Review Questions 19 and 20.

Q: How do you evaluate a numerical expression that involves zero or negative exponents?

A: Any non-zero number raised to the exponent 0 equals 1: $a^0 = 1$ for $a \neq 0$.

Any non-zero number raised to a negative exponent equals the reciprocal of the number raised to the opposite exponent: $a^{-n} = \frac{1}{a^n}$ for $a \neq 0$.

EXAMPLE

Evaluate.

a) 4^0 **b)** 6^{-2}

Solution

a) $4^0 = 1$

b) $6^{-2} = \frac{1}{6^2}$
$= \frac{1}{36}$

PRACTICE Questions

Lesson 3.1

1. State whether each relation is quadratic. Justify your decision.

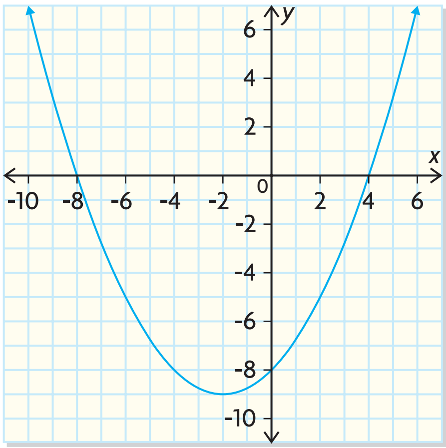
a) $y = 4x - 5$

b)

x	−3	−2	−1	0	1	2	3
y	56	35	18	5	−4	−9	−10

c) $y = 2x(x - 5)$

d)



2. Discuss how the graph of the quadratic relation $y = ax^2 + bx + c$ changes as a, b, and c are changed.

Lesson 3.2

3. Graph each quadratic relation and determine

i) the equation of the axis of symmetry

ii) the coordinates of the vertex

iii) the y-intercept

iv) the zeros

a) $y = x^2 - 8x$ **b)** $y = x^2 + 2x - 15$

4. Verify your results for question 3 using graphing technology.

5. The x-intercepts of a quadratic relation are −2 and 5, and the second differences are negative.

a) Is the y-value of the vertex a maximum value or a minimum value? Explain.

b) Is the y-value of the vertex positive or negative? Explain.

c) Calculate the x-coordinate of the vertex.

6. Create tables of values for three parabolas that go through the point (2, 7). How do you know that each table of values represents a parabola?

7. Use graphing technology to graph the parabola for each relation below. Then determine

i) the x-intercepts

ii) the equation of the axis of symmetry

iii) the coordinates of the vertex

a) $y = -x^2 + 18x$

b) $y = 6x^2 + 15x$

8. What does a in the equation $y = ax^2 + bx + c$ tell you about the parabola?

9. The Rudy Snow Company makes custom snowboards. The company's profit can be modelled with the relation $y = -6x^2 + 42x - 60$, where x is the number of snowboards sold (in thousands) and y is the profit (in hundreds of thousands of dollars).

a) How many snowboards does the company need to sell to break even?

b) How many snowboards does the company need to sell to maximize their profit?

Lesson 3.3

10. The x-intercepts of a parabola are −2 and 7, and the y-intercept is −28.

a) Determine an equation for the parabola.

b) Determine the coordinates of the vertex.

11. Determine an equation for each parabola.

a) The x-intercepts are 5 and 9, and the y-coordinate of the vertex is −2.

b) The x-intercepts are −3 and 7, and the y-coordinate of the vertex is 4.

c) The x-intercepts are −6 and 2, and the y-intercept is −9.

d) The vertex is (4, 0), and the y-intercept is 8.

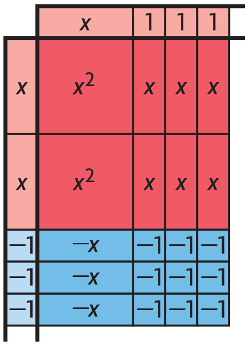
e) The x-intercepts are −3 and 3, and the parabola passes through the point (2, 20).

12. A bus company usually transports 12 000 people per day at a ticket price of \$1. The company wants to raise the ticket price. For every \$0.10 increase in the ticket price, the number of riders per day is expected to decrease by 400. Calculate the ticket price that will maximize revenue.

Lesson 3.4

13. Identify the binomial factors and their products.

a)



b)

	$5x$	-6
$3x$	$15x^2$	$-18x$
-4	$-20x$	24

14. Expand and simplify.

a) $(x + 5)(x + 4)$

b) $(x - 2)(x - 5)$

c) $(2x - 3)(2x + 3)$

d) $(4x + 5)(3x - 2)$

e) $(4x - 2y)(5x + 3y)$

f) $(6x - 2)(5x + 7)$

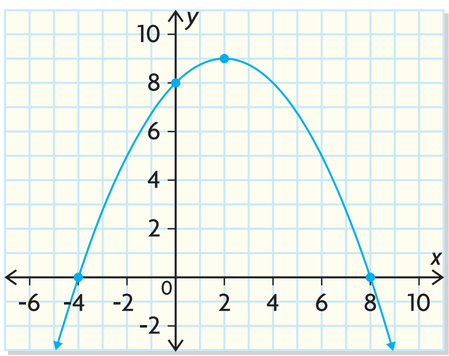
15. Expand and simplify.

a) $(2x + 6)^2$

b) $-2(-2x + 5)(3x + 4)$

c) $2x(4x - y)(4x + y)$

16. Determine the equation of the parabola. Express your answer in standard form.



Lesson 3.5

17. A model rocket is shot straight up into the air. The table shows its height, y, in metres after x seconds.

Time (s)	0	1	2	3	4	5	6
Height (m)	0.0	25.1	40.4	45.9	41.6	27.5	3.6

a) Sketch a curve of good fit.

b) Is the curve of good fit a parabola? Explain.

c) Determine the equation of your curve of good fit. Express your answer in standard form.

d) Estimate the height of the rocket after 4.5 s.

e) When is the rocket at a height of 20 m?

18. A sandbag is dropped into the ocean from a hot air balloon to make the balloon rise. The table shows the height of the sandbag at different times as it falls.

Time (s)	0	2	4	6	8	10
Height (m)	1200	1180	1120	1020	880	700

a) Draw a scatter plot of the data.

b) Sketch a curve of good fit.

c) Is the curve of good fit a parabola? Explain.

d) Determine the equation of your curve of good fit. Express your answer in standard form.

e) Estimate the time when the sandbag hits the water.

Lesson 3.6

19. Evaluate. Express your answers in rational form.

a) 2^{-3}

b) -5^{-1}

c) $\left(\frac{2}{5}\right)^{-2}$

d) $(-9)^0$

e) 4^{-3}

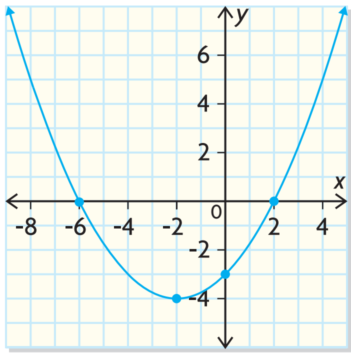
f) $-\left(\frac{1}{6}\right)^{-2}$

20. Which do you think is greater: $\left(\frac{1}{4}\right)^2$ or 3^{-2}? Justify your decision.

21. For what postive values of x is x^2 greater than 2^x? How do you know?

3 Chapter Self-Test

1. State the zeros, vertex, and equation of the axis of symmetry of the parabola at the right.



2. The points $(-9, 0)$ and $(19, 0)$ lie on a parabola.
   a) Determine an equation for its axis of symmetry.
   b) The y-coordinate of the vertex is -28. Determine an equation for the parabola in factored form.
   c) Write your equation for part b) in standard form.

3. Decide, without graphing, whether each data set can be modelled by a quadratic relation. Explain how you made your decision.

a)

x	−1	0	1	2	3
y	1	2	−3	−14	−31

b)

x	0	1	2	3	4
y	−4	−3	0	5	12

4. Sketch each graph. Label the intercepts and the vertex using their coordinates.
   a) $y = (x - 6)(x + 2)$ **b)** $y = -(x - 6)(x + 4)$

5. The population, P, of a city is modelled by the equation $P = 14t^2 + 820t + 42\,000$, where t is the time in years. When $t = 0$, the year is 2008.
   a) Determine the population in 2018.
   b) When was the population about 30 000?

6. Expand and simplify.
   a) $(2x - 3)(5x + 2)$ **b)** $(3x - 4y)(5x + 2y)$ **c)** $-5(x - 4)^2$

7. A toy rocket is placed on a tower and launched straight up. The table shows its height, y, in metres above the ground after x seconds.

Time, x (s)	0	1	2	3	4	5	6	7	8
Height, y (m)	16	49	72	85	88	81	64	37	0

   a) What is the height of the tower?
   b) How long is the rocket in flight?
   c) Do the data in the table represent a quadratic relation? Explain.
   d) Create a scatter plot. Then draw a curve of good fit.
   e) Determine the equation of your curve of good fit.
   f) What is the maximum height of the rocket?

8. In what ways is modelling a problem using a quadratic relation similar to using a linear relation? In what ways is it different?

9. Evaluate.
   a) 7^{-2} **b)** -3^0 **c)** $-\left(\frac{2}{3}\right)^{-4}$ **d)** -5^{-3}

Process Checklist

✔ Question 2: Did you relate the characteristics of the graphical **representation** of the relation with its equation?

✔ Questions 5 and 7: Did you select appropriate **problem solving** strategies for each situation?

✔ Question 8: Did you make **connections** to **communicate** a variety of ways to relate modelling with linear and quadratic relations?

3 Chapter Task

Comparing the Force of Gravity

Gravity is the force of attraction between two objects. This force varies in our solar system. For example, the Moon's diameter is about one-fourth of Earth's diameter, so the Moon's gravity is much less than Earth's.

A measure of the strength of gravity is the value *g*, which is the acceleration (or rate of change of velocity) of a freely falling object. One strategy for calculating *g* is to drop an object and time its fall to the ground. These tables show the time that an object takes to fall from a height of 10 m on both Earth and the Moon.

Earth

Time (s)	0.0	0.2	0.4	0.6	0.8	1.0	1.2	1.4
Height (m)	10.0000	9.8038	9.2152	8.2342	6.8608	5.0950	2.9368	0.3862

Moon

Time (s)	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5
Height (m)	10.0000	9.7975	9.1900	8.1775	6.7600	4.9375	2.7100	0.0775

? What factor relates the values of *g* on the Moon and Earth?

A. Create a scatter plot for each data set and draw a curve of good fit.

B. Do the data show that the relation between time and height is quadratic on Earth or the Moon? Explain.

C. Estimate the location of the vertex, the axis of symmetry, and the zeros.

D. Use a strategy of your choice to determine the equations of curves of good fit in standard form.

E. An equation that models this situation is $y = H - \left(\frac{g}{2}\right)x^2$, where H is the initial height of the object above the ground and y is the height of the object at time x. If x is measured in seconds, and y and H are measured in metres, then the units for g are metres per square second (m/s^2). Use this information and the equations you determined for part D to calculate the values of g on Earth and the Moon.

F. Calculate the factor that relates the value of *g* on the Moon to the value of *g* on Earth.

Task **Checklist**

- ✔ Did you label your graphs?
- ✔ Did you include your calculations?
- ✔ Did you explain your thinking clearly?
- ✔ Did you calculate the factor that relates the value of *g* on the Moon to the value of *g* on Earth?