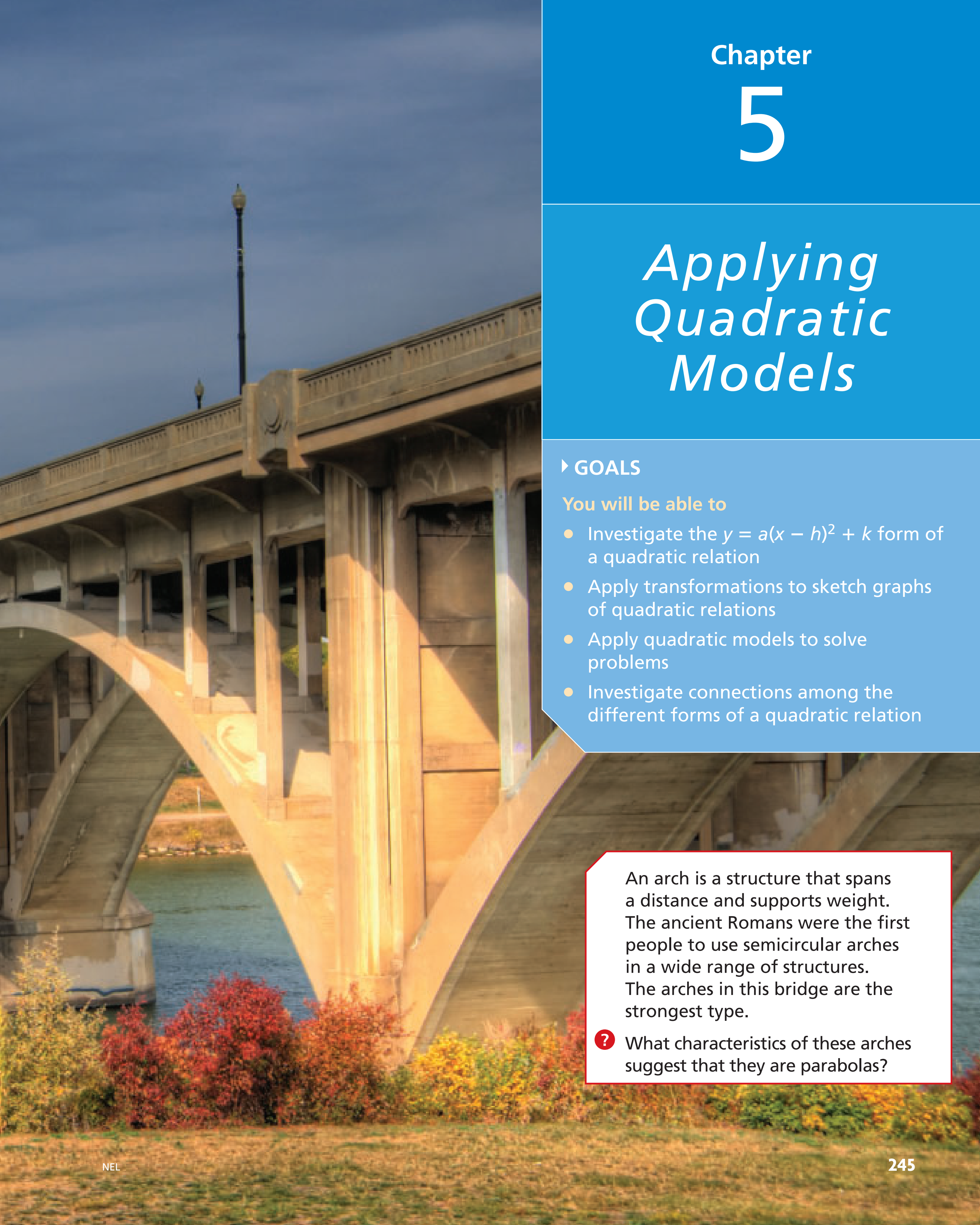


Chapter

5

Applying Quadratic Models

GOALS

You will be able to

- Investigate the $y = a(x - h)^2 + k$ form of a quadratic relation
- Apply transformations to sketch graphs of quadratic relations
- Apply quadratic models to solve problems
- Investigate connections among the different forms of a quadratic relation

An arch is a structure that spans a distance and supports weight. The ancient Romans were the first people to use semicircular arches in a wide range of structures. The arches in this bridge are the strongest type.

? What characteristics of these arches suggest that they are parabolas?

5 Getting Started

WORDS YOU NEED to Know

1. Match each term at the left with the correct description or example.

a) transformation
b) translation
c) reflection
d) parabola
e) vertex
f) factored form of a quadratic relation

i) a point that relates to the maximum or minimum value of a quadratic relation
ii) the result of moving or changing the size of a shape according to a rule
iii) the result of sliding each point on a shape the same distance in the same direction
iv) $y = a(x - r)(x - s)$
v) the result of flipping a shape to produce a mirror image of the shape
vi) the graph of a quadratic relation

SKILLS AND CONCEPTS You Need

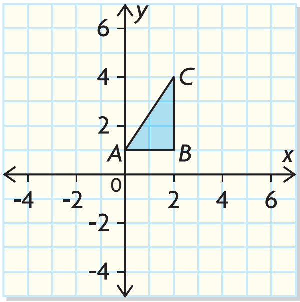
Working with Transformations

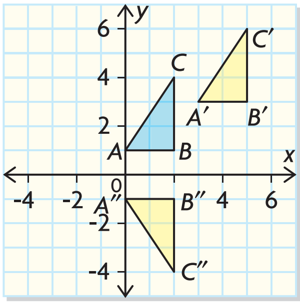
Study Aid

- For more help and practice, see Appendix A-13.

Translations, **reflections**, **rotations**, and **dilatations** are types of transformations. They can be applied to a point, a line, or a figure.

EXAMPLE





Apply the following transformations to $\triangle ABC$ shown at the left.

a) Translate $\triangle ABC$ 3 units right and 2 units up.

b) Reflect $\triangle ABC$ in the x-axis.

Solution

Apply the same translation and the same reflection to points A, B, and C. Plot the image points, and draw each image triangle.

a)

Original Point	Image Point
$A(0, 1)$	$A'(3, 3)$
$B(2, 1)$	$B'(5, 3)$
$C(2, 4)$	$C'(5, 6)$

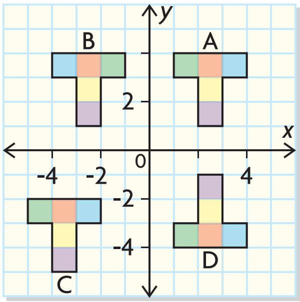
b)

Original Point	Image Point
$A(0, 1)$	$A''(0, -1)$
$B(2, 1)$	$B''(2, -1)$
$C(2, 4)$	$C''(2, -4)$

2. a) In the diagram at the right, is figure B, figure C, or figure D the result of a translation of figure A? Explain.

b) Which figure is the result of a reflection of figure A in the x-axis? Explain.

c) Which figure is the result of a reflection of figure A in the y-axis? Explain.



Understanding Quadratic Relations

A quadratic relation can be expressed algebraically as an equation in standard form or factored form. You can determine information about its parabola from the equation of the relation.

Study Aid

- For more help and practice, see Lesson 3.3, Examples 2 to 4.

EXAMPLE

Determine the properties of the relation defined by the equation $y = (x - 2)(x - 4)$. Then sketch the graph of the relation.

Solution

The equation $y = (x - 2)(x - 4)$ is the equation of a quadratic relation in factored form.

The values $x = 2$ and $x = 4$ make the factors $(x - 2)$ and $(x - 4)$ equal to zero. These are called the zeros of the relation and are the x-intercepts of the graph.

The axis of symmetry is the perpendicular bisector of the line segment that joins the zeros. Its equation is $x = \frac{2 + 4}{2}$ or $x = 3$.

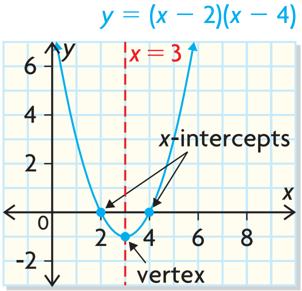
The vertex of the parabola lies on the axis of symmetry. To determine the y-coordinate of the vertex, substitute $x = 3$ into the equation and evaluate.

$y = (3 - 2)(3 - 4)$

$y = (1)(-1)$

$y = -1$

The vertex is $(3, -1)$.



3. Determine the value of y in each quadratic relation for each value of x.

a) $y = x^2 + 2x + 5$, when $x = -4$

b) $y = x^2 - 3x - 28$, when $x = 7$

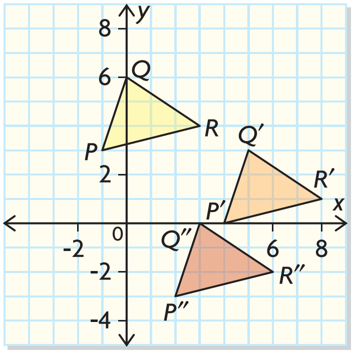
4. Determine the zeros, the equation of the axis of symmetry, and the vertex of each quadratic relation.

a) $y = (x + 5)(x - 3)$ **b)** $y = 2(x - 4)(x + 1)$ **c)** $y = -4x(x + 3)$

Study | ***Aid***

- For help, see the Review of Essential Skills and Knowledge Appendix.

Question	Appendix
5, 6	A-13



PRACTICE

5. State the coordinates of the image point after applying the indicated transformation(s).

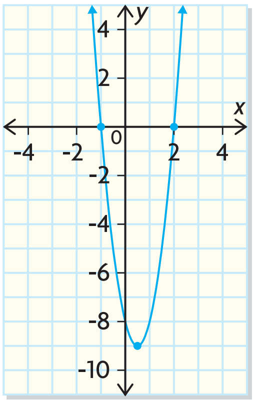
a) $A(3, 4)$ is translated 2 units left and 5 units up.
b) $B(-1, -5)$ is translated 4 units right and 3 units down.
c) $C(2, -7)$ is translated 2 units left and 7 units up.
d) $D(3, -5)$ is reflected in the x-axis.

6. Describe the transformation that was used to translate each triangle onto the image in the diagram at the left.

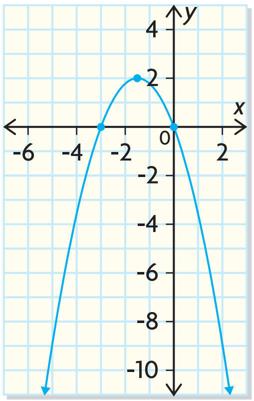
a) $\triangle PQR$ to $\triangle P'Q'R'$
b) $\triangle PQR$ to $\triangle P''Q''R''$
c) $\triangle P'Q'R'$ to $\triangle P''Q''R''$

7. State the zeros, equation of the axis of symmetry, and the vertex of each quadratic relation.

a)



b)



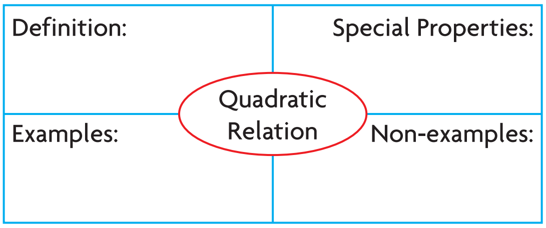
8. Express each quadratic relation in standard form.

a) $y = (x + 4)(x + 5)$
b) $y = (2x - 3)(x + 2)$
c) $y = -3(x - 4)(x + 7)$
d) $y = (x + 5)^2$

9. Sketch the graph of each quadratic relation in question 8.

10. Express $y = 2x^2 - 4x - 48$ in factored form. Then determine its minimum value.

11. Copy and complete the chart to show what you know about quadratic relations. Share your chart with a classmate.



APPLYING *What You Know*

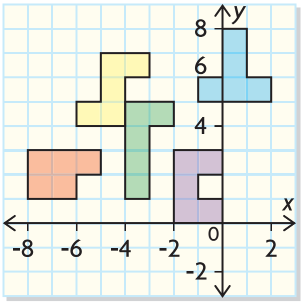
Tiling Transformers

Jesse and Tyler decided to have a competition to see who could transform the figures from the Start grid to the End grid in the fewest number of moves.

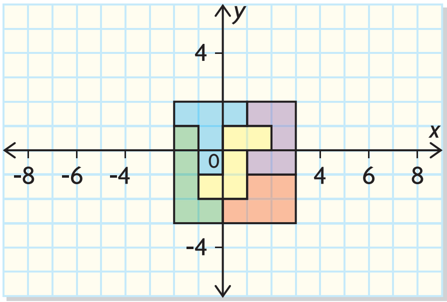
YOU WILL NEED

- grid paper
- ruler
- coloured pencils or markers
- scissors

Start



End



? What is the fewest number of moves needed to transform these figures?

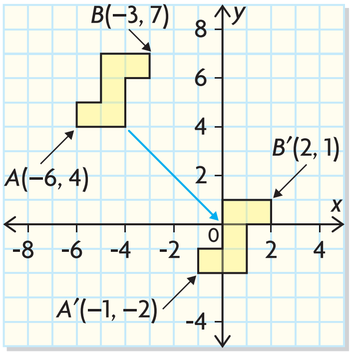
A. On a piece of grid paper, draw the five figures on the Start grid. Cut out each figure.

B. Draw *x*- and *y*-axes on another piece of grid paper. Label each axis from −10 to 10.

C. Put the yellow figure on the grid paper in the position shown on the Start grid.

D. Apply one or more transformations to move the yellow figure to the position shown on the End grid.

E. Make a table like the one below. Record the transformation(s) that you used to move the yellow figure for part D.



Player 1				
Move	**Figure**	**Original Coordinates**	**Transformation**	**New Coordinates**
1	yellow	*A*(–6, 4), *B*(–3, 7)	translation 5 units right, 6 units down	*A*′(–1, –2), *B*′(2, 1)
2				

F. Repeat parts C to E for another figure. (The second figure can move across the first figure if necessary.)

G. Continue transforming figures and recording results until you have created the design shown on the End grid.

H. How many moves did you need? Compare your results with those of other classmates. What is the fewest number of moves needed?

5.1

Stretching/Reflecting Quadratic Relations

YOU WILL NEED

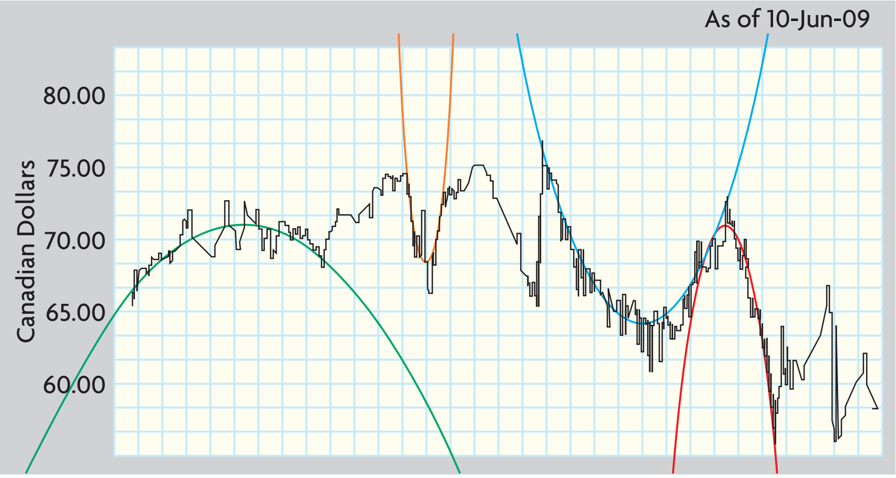
- graphing calculator
- dynamic geometry software, or grid paper and ruler

GOAL

Examine the effect of the parameter *a* in the equation $y = ax^2$ on the graph of the equation.

INVESTIGATE the Math

Suzanne's mother checks the family's investments regularly. When Suzanne saw the stock chart that her mother was checking, she noticed trends in sections of the graph. These trends looked like the shapes of the parabolas she had been studying. Each "parabola" was a different shape.



? What is the relationship between the value of *a* in the equation $y = ax^2$ and the shape of the graph of the relation?

A. Enter $y = x^2$ as Y1 in the equation editor of a graphing calculator.

B. The window settings shown are "friendly" because they allow you to trace using intervals of 0.1. Graph the parabola using these settings.

```
WINDOW
 Xmin=-4.7
 Xmax=4.7
 Xscl=1
 Ymin=-3.1
 Ymax=9.3
 Yscl=1
 Xres=1
```

C. Enter $y = 2x^2$ in Y2 and $y = 5x^2$ in Y3, and graph these quadratic relations. What appears to be happening to the shape of the graph as the value of *a* increases?

Tech *Support*

For help graphing relations and adjusting the window settings using a TI-83/84 graphing calculator, see Appendix B-2 and B-4. If you are using a TI-*n*spire, see Appendix B-38 and B-40.

D. Where would you expect the graph of $y = 3x^2$ to appear, relative to the other three graphs? Check by entering $y = 3x^2$ into Y4 and graph with a thick line. Was your conjecture correct?

E. Where would you expect the graphs of $y = \frac{1}{2}x^2$ and $y = \frac{1}{4}x^2$ to appear, relative to the graph of $y = x^2$? Clear the equations from Y2, Y3, and Y4. Enter $y = \frac{1}{2}x^2$ into Y2 and $y = \frac{1}{4}x^2$ into Y3, and graph these quadratic relations. Describe the effect of the **parameter** a on the parabola when $0 < a < 1$.

F. Where you would expect the graph of $y = \frac{3}{4}x^2$ to appear, relative to the other three graphs? Check by entering $y = \frac{3}{4}x^2$ into Y4 and graph with a thick line.

G. Clear the equations from Y2, Y3, and Y4. Enter $y = -4x^2$ into Y2 and $y = -\frac{1}{4}x^2$ into Y3, and graph these quadratic relations. Describe the effect of a on the parabola when $a < 0$.

H. Ask a classmate to give you an equation in the form $y = ax^2$, where $a < 0$. Describe to your classmate what its graph would look like relative to the other three graphs. Verify your description by graphing the equation in Y4.

I. How does changing the value of a in the equation $y = ax^2$ affect the shape of the graph?

Reflecting

J. Which parabola in the stock chart has the greatest value of a? Which has the least value of a? Which parabolas have negative values of a? Explain how you know.

K. What happens to the x-coordinates of all the points on the graph of $y = x^2$ when the parameter a is changed in $y = ax^2$? What happens to the y-coordinates? What happens to the shape of the parabola near its vertex?

L. State the ranges of values of a that will cause the graph of $y = x^2$ to be
i) **vertically stretched**
ii) **vertically compressed**
iii) reflected across the x-axis

Tech Support

Move the cursor to the left of Y4. Press ENTER to change the line style to make the line thick.

parameter

a coefficient that can be changed in a relation; for example, a, b, and c are parameters in $y = ax^2 + bx + c$

vertical stretch

a transformation that increases all the y-coordinates of a relation by the same factor

vertical compression

a transformation that decreases all the y-coordinates of a relation by the same factor

APPLY the Math

EXAMPLE 1 Selecting a transformation strategy to graph a parabola

a) Sketch the graph of the equation $y = 3x^2$ by transforming the graph of $y = x^2$.

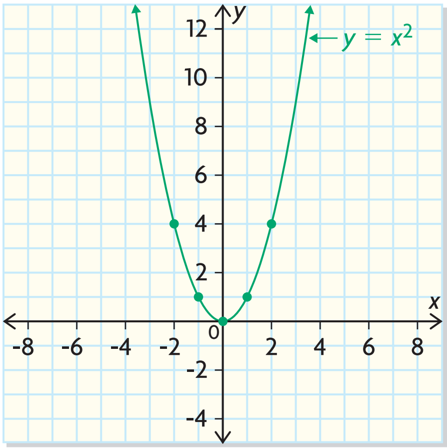
b) Describe how the graphs of $y = 3x^2$ and $y = -3x^2$ are related.

Zack's Solution

a)

x	-2	-1	0	1	2
y	4	1	0	1	4

I created a table of values to determine five points on the graph of $y = x^2$.



I plotted the points on a grid and joined them with a smooth curve.

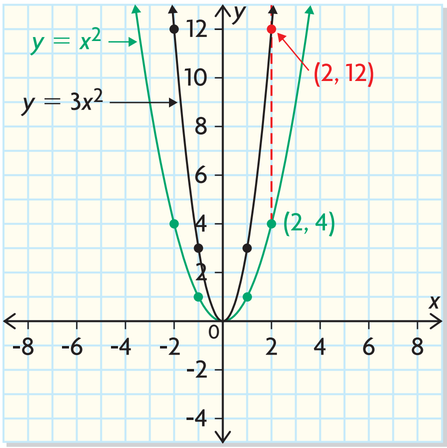
I can use these five points any time I want to sketch the graph of $y = x^2$ because they include the vertex and two points on each side of the parabola.

I decided to call this my five-point sketch.

x	-2	-1	0	1	2
y	12	3	0	3	12

To transform my graph into a graph of $y = \mathbf{3}x^2$, I multiplied the y-coordinates of each point on $y = x^2$ by **3**. For example,

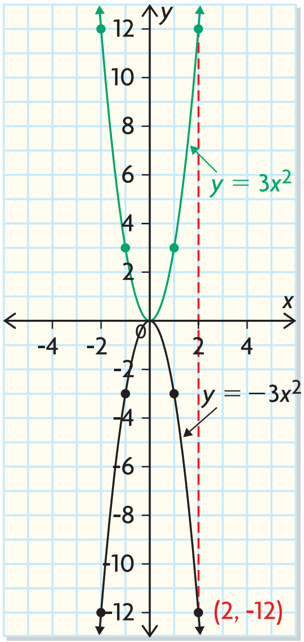
$$(2, 4) \searrow 4 \times \mathbf{3} \nearrow (2, \mathbf{12})$$



I plotted and joined my new points to get the graph of $y = 3x^2$. $a = 3$ represents a vertical stretch by a factor of 3. This means that the y-coordinates of the points on the graph of $y = 3x^2$ will become greater faster, so the parabola will be narrower near its vertex compared to the graph of $y = x^2$.

b)

x	-2	-1	0	1	2
y	-12	-3	0	-3	-12



To get the graph of $y = -3x^2$, I multiplied the y-coordinates of all the points on the graph of $y = 3x^2$ by $\mathbf{-1}$. For example,

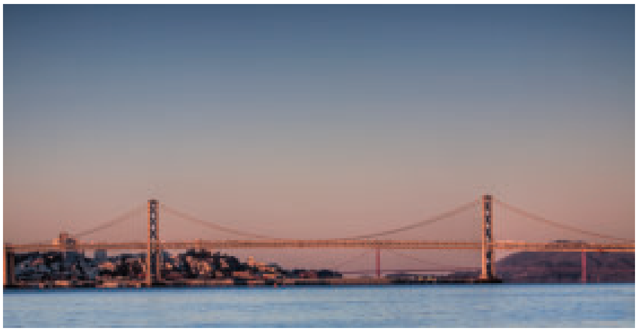
$$(2, 12) \searrow 12 \times (\mathbf{-1}) \nearrow (2, \mathbf{-12})$$

$a = -3$ represents a vertical stretch by a factor of 3 and a reflection in the x-axis. This means that all the points on the graph of $y = 3x^2$ are reflected in the x-axis.

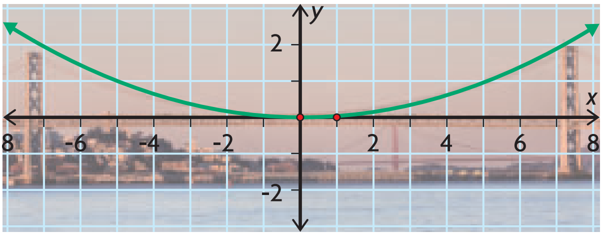
The graph of $y = -3x^2$ is the reflection of the graph of $y = 3x^2$ in the x-axis.

EXAMPLE 2 **Connecting** the value of *a* to a graph

Determine an equation of a quadratic relation that models the arch of San Francisco's Bay Bridge in the photograph below.



Mary's Solution: Representing the picture on a hand-drawn grid



I located a point on the graph and estimated the coordinates of the point to be (5, 1).

I used a photocopy of the photograph. I laid a transparent grid with axes on top of the photocopy.

I placed the origin at the vertex of the arch. I did this since all parabolas defined by $y = ax^2$ have their vertex at (0, 0).

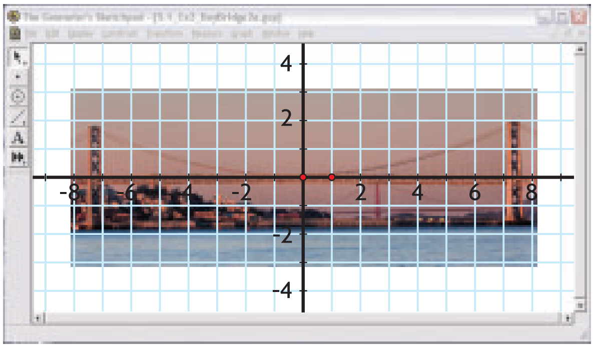
$$\begin{aligned} y &= ax^2 \\ 1 &= a(5)^2 \\ 1 &= 25a \\ \frac{1}{25} &= \frac{25a}{25} \\ \frac{1}{25} &= a \end{aligned}$$

The equation of the graph is in the form $y = ax^2$. To determine the value of a, I had to determine the coordinates of a point on the parabola. I chose the point (5, 1). I substituted $x = 5$ and $y = 1$ into the equation and solved for a.

An equation that models the arch of the bridge is $y = \frac{1}{25}x^2$.

The graph that models the arch is a vertical compression of the graph of $y = x^2$ by a factor of $\frac{1}{25}$.

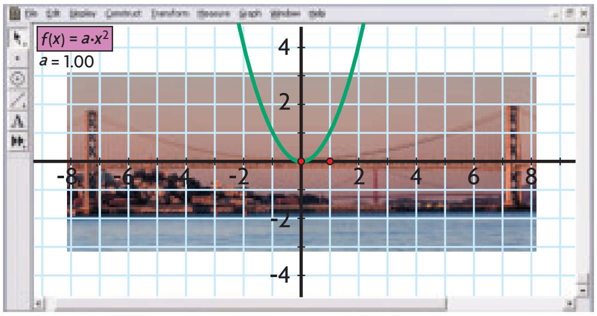
Sandeep's Solution: Selecting dynamic geometry software



I imported the photograph into dynamic geometry software. I superimposed a grid over the photograph. Then I adjusted the grid so that the origin was at the vertex of the bridge's parabolic arch. I need to create a graph using the relation $y = ax^2$ by choosing a value for the parameter a.

Tech | Support

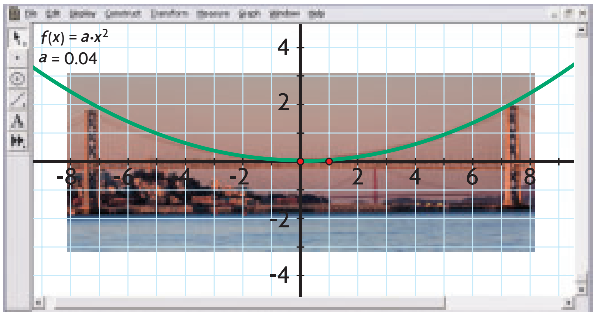
For help creating and graphing relations using parameters in dynamic geometry software, as well as animating the parameter, see Appendix B-17.



$f(x) = a \cdot x^2$

$a = 1.00$

When I used $a = 1$, the graph of $y = x^2$ appeared. The parabola was too narrow. It had to be vertically compressed to fit the arch. To do this, I needed a lower value of a, between 0 and 1. I needed a positive value because the arch opens upward.



$f(x) = a \cdot x^2$

$a = 0.04$

I tried $a = 0.5$, but the parabola was not wide enough.

I tried $a = 0.1$. This value gave me a better fit. I still wasn't satisfied, so I tried different values of a between 0 and 0.1. I found that $a = 0.04$ gave me a good fit.

An equation that models the bridge is $y = 0.04x^2$.

Vertically compressing the graph of $y = x^2$ by a factor of 0.04 creates a graph that fits the photograph.

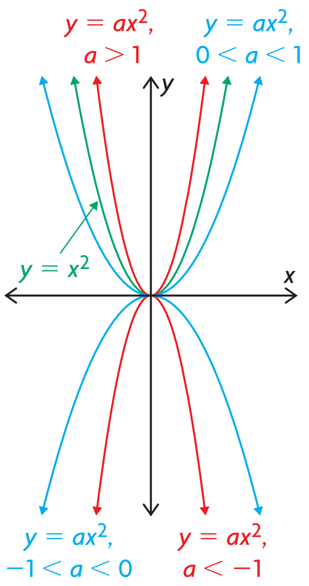
In Summary

Key Idea

- When compared with the graph of $y = x^2$, the graph of $y = ax^2$ is a parabola that has been stretched or compressed vertically by a factor of a.

Need to Know

- Vertical stretches are determined by the value of a. When $a > 1$, the graph is stretched vertically. When $a < -1$, the graph is stretched vertically and reflected across the x-axis.
- Vertical compressions are also determined by the value of a. When $0 < a < 1$, the graph is compressed vertically. When $-1 < a < 0$, the graph is compressed vertically and reflected across the x-axis.
- If $a > 0$, the parabola opens upward.
- If $a < 0$, the parabola opens downward.

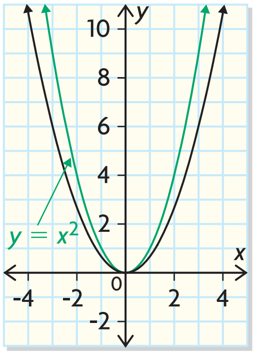


CHECK Your Understanding

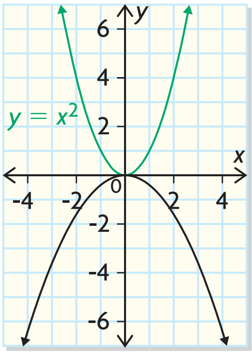
1. Match each graph with the correct equation. The graph of $y = x^2$ is in green in each diagram.

   a) $y = 4x^2$
   b) $y = -3x^2$
   c) $y = \frac{2}{3}x^2$
   d) $y = -0.4x^2$

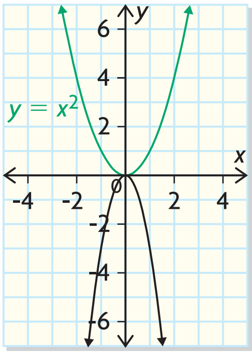
i)



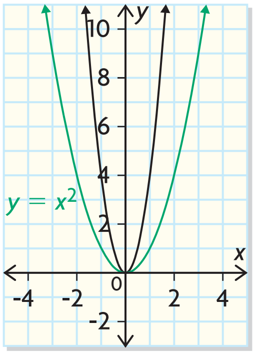
ii)



iii)



iv)



2. The graph of $y = x^2$ is transformed to $y = ax^2$ $(a \neq 1)$. For each point on $y = x^2$, determine the coordinates of the transformed point for the indicated value of a.

   a) $(1, 1)$, when $a = 5$
   b) $(-2, 4)$, when $a = -3$
   c) $(5, 25)$, when $a = -0.6$
   d) $(-4, 16)$, when $a = \frac{1}{2}$

3. Write the equations of two different quadratic relations that match each description.

   a) The graph is narrower than the graph of $y = x^2$ near its vertex.
   b) The graph is wider than the graph of $y = -x^2$ near its vertex.
   c) The graph opens downward and is narrower than the graph of $y = 3x^2$ near its vertex.

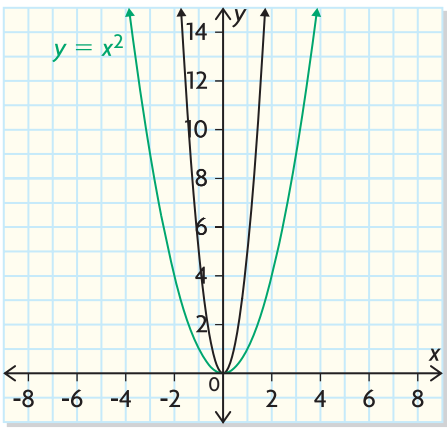
PRACTISING

4. K Sketch the graph of each equation by applying a transformation to the graph of $y = x^2$. Use a separate grid for each equation, and start by sketching the graph of $y = x^2$.

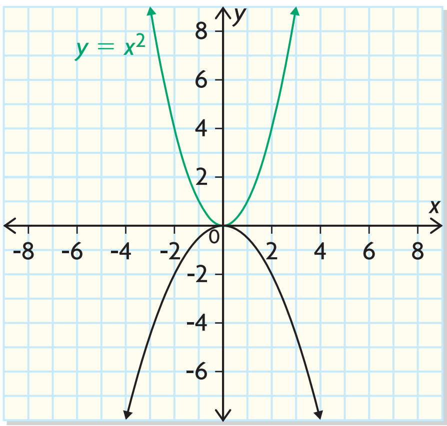
   a) $y = 3x^2$
   b) $y = -0.5x^2$
   c) $y = -2x^2$
   d) $y = \frac{1}{4}x^2$
   e) $y = -\frac{3}{2}x^2$
   f) $y = 5x^2$

5. Describe the transformation(s) that were applied to the graph of $y = x^2$ to obtain each black graph. Write the equation of the **black** graph.

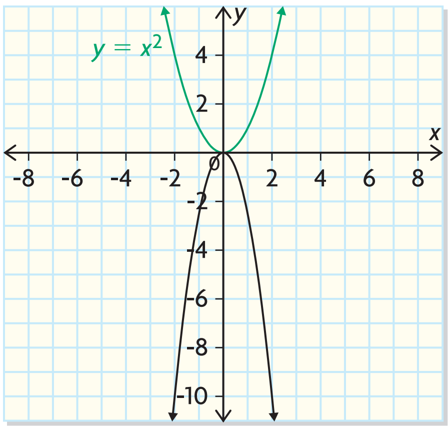
a)



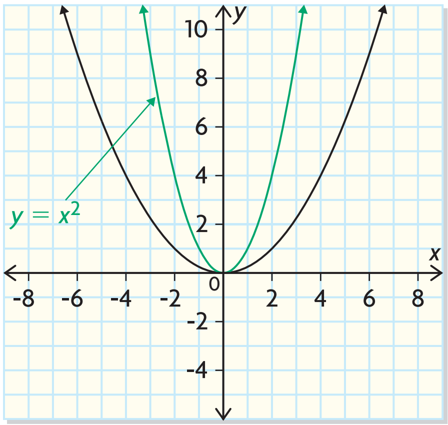
b)



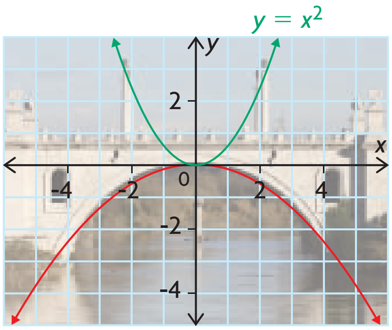
c)



d)

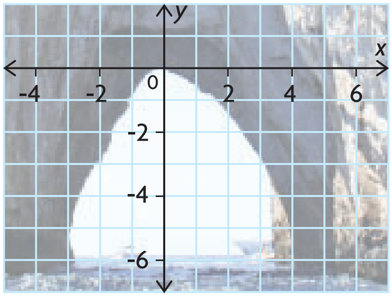


6. Andy modelled the arch of the bridge in the photograph at the right by tracing a parabola onto a grid. Now he wants to determine an equation of the parabola. Explain the steps he should use to do this, and state the equation.

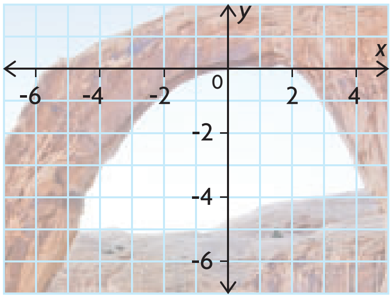


7. Determine an equation of a quadratic model for each natural arch.

a) Isle of Capri in Italy

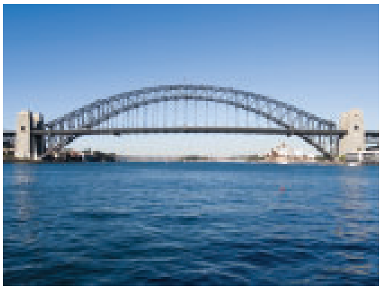


b) Corona Arch in Utah



8. Identify the transformation(s) that must be applied to the graph of $y = x^2$ to create a graph of each equation. Then state the coordinates of the image of the point (2, 4).

a) $y = 4x^2$
b) $y = -\frac{2}{3}x^2$
c) $y = 0.25x^2$
d) $y = -5x^2$
e) $y = -x^2$
f) $y = \frac{1}{5}x^2$



9. **A** By tracing the bridge at the left onto a grid, determine an equation that models the lower outline of the Sydney Harbour Bridge in Australia.

10. **T** Seth claims that changing the value of a in quadratic relations of the form $y = ax^2$ will never result in a parabola that is congruent to the parabola $y = x^2$. Do you agree or disagree? Justify your decision.

11. Copy and complete the following table.

Equation	Direction of Opening (upward/ downward)	Description of Transformation (stretch/ compress)	Shape of Graph Compared with Graph of $y = x^2$ (wider/narrower)
$y = 5x^2$			
$y = 0.25x^2$			
$y = -\frac{1}{3}x^2$			
$y = -8x^2$			

12. Explain why it makes sense that each statement about the graph of $y = ax^2$ is true.

a) If $a < 0$, then the parabola opens downward.
b) If a is a rational number between -1 and 1, then the parabola is wider than the graph of $y = x^2$.
c) The vertex is always (0, 0).

Extending

13. The graph of $y = ax^2$ ($a \neq 1$, $a > 0$) is either a vertical stretch or a vertical compression of the graph of $y = x^2$. Use graphing technology to determine whether changing the value of a has a similar effect on the graphs of equations such as $y = ax$, $y = ax^3$, $y = ax^4$, and $y = ax^{\frac{1}{2}}$.

14. The equation of a circle with radius r and centre (0, 0) is $x^2 + y^2 = r^2$.

a) Explore the effect of changing positive values of a when graphing $ax^2 + ay^2 = r^2$.
b) Explore the effects of changing positive values of a and b when graphing $ax^2 + by^2 = r^2$.

5.2

Exploring Translations of Quadratic Relations

GOAL

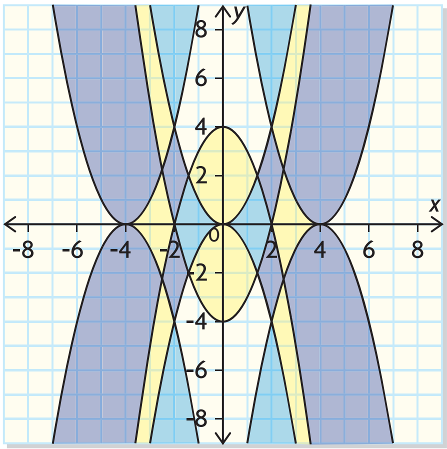
Investigate the roles of h and k in the graphs of $y = x^2 + k$, $y = (x - h)^2$, and $y = (x - h)^2 + k$.

YOU WILL NEED

- grid paper
- ruler
- graphing calculator

EXPLORE the Math

Hammad has been asked to paint a mural of overlapping parabolas on a wall in his school. A sketch of his final design is shown at the right. He is using his graphing calculator to try to duplicate his design. His design uses parabolas that have the same shape as $y = x^2$, but he doesn't know what equations he should enter into his graphing calculator to place the parabolas in different locations on the screen.



? What is the connection between the location of the vertex of a parabola and the equation of its quadratic relation?

A. Enter the equation $y = x^2$ as Y1 in the equation editor of a graphing calculator. Graph the equation using the window settings shown.

```
WINDOW
 Xmin=-10
 Xmax=10
 Xscl=1
 Ymin=-10
 Ymax=10
 Yscl=1
 Xres=1
```

B. Enter an equation of the form $y = x^2 + k$ in Y2 by adding or subtracting a number after the x^2 term. For example, $y = x^2 + 1$ or $y = x^2 - 3$. Graph your equation, and compare the graph with the graph of $y = x^2$. Try several other equations, replacing the one you have in Y2 each time. Be sure to change the number you add or subtract after the x^2 term.

C. Copy this table. Use the table to record your findings for part B.

Value of k	Equation	Distance and Direction from $y = x^2$	Vertex
0	$y = x^2$	not applicable	(0, 0)

Tech *Support*

For help graphing relations, changing window settings, and tracing along a graph using a TI-83/84 graphing calculator, see Appendix B-2 and B-4. If you are using a TI-*n*spire, see Appendix B-38 and B-40.

Tech *Support*

Use the TRACE key and the up arrow ▲ to help you distinguish one graph from another.

D. Investigate what happens to the graph of $y = x^2$ when a number is added to or subtracted from the value of x before it is squared, creating an equation of the form $y = (x - h)^2$. For example, $y = (x + 1)^2$ or $y = (x - 2)^2$. Graph your new equations in Y2 each time using a graphing calculator. Then copy this table and record your findings.

Value of h	Equation	Distance and Direction from $y = x^2$	Vertex
0	$y = x^2$	not applicable	(0, 0)

E. Identify the type of transformations that have been applied to the graph of $y = x^2$ to obtain the graphs in your table for part C and your table for part D.

F. Make a conjecture about how you could predict the equation of a parabola if you knew the translations that were applied to the graph of $y = x^2$.

G. Copy and complete this table to investigate and test your conjecture for part F.

Value of h	Value of k	Equation	Relationship to $y = x^2$		Vertex
			Left/Right	Up/Down	
0	0	$y = x^2$	not applicable	not applicable	(0, 0)
			left 3	down 5	
4	1	$y = (x - 4)^2 + 1$			
					(−2, 6)
		$y = (x + 5)^2 - 3$			

H. Use what you have discovered to identify the equations that Hammad should type into his calculator to graph the parabolas in the mural design.

I. If the equation of a quadratic relation is given in the form $y = (x - h)^2 + k$, what can you conclude about its vertex?

Reflecting

J. Describe how changing the value of k in $y = x^2 + k$ affects
- **i)** the graph of $y = x^2$
- **ii)** the coordinates of each point on the parabola $y = x^2$
- **iii)** the parabola's vertex and axis of symmetry

K. Describe how changing the value of h in $y = (x - h)^2$ affects
- **i)** the graph of $y = x^2$
- **ii)** the coordinates of each point on the parabola $y = x^2$
- **iii)** the parabola's vertex and axis of symmetry

L. For parabolas defined by $y = (x - h)^2 + k$,

i) how do their shapes compare to the parabola defined by $y = x^2$?

ii) what is the equation of the axis of symmetry?

iii) what are the coordinates of the vertex?

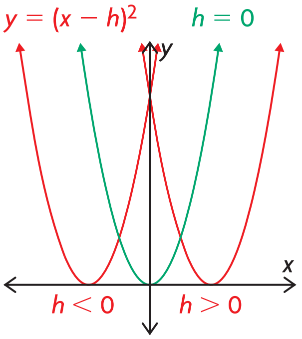
In Summary

Key Ideas

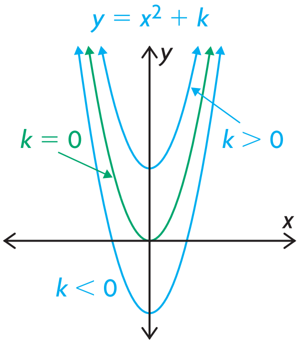
- The graph of $y = (x - h)^2 + k$ is congruent to the graph of $y = x^2$, but translated horizontally and vertically.
- Translations can also be described as shifts. Vertical shifts are up or down, and horizontal shifts are left or right.

Need to Know

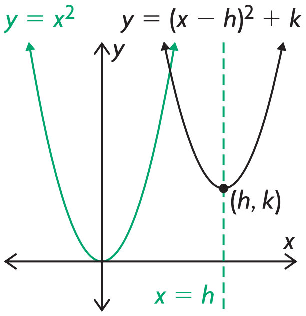
- The value of h tells how far and in what direction the parabola is translated horizontally. If $h < 0$, the parabola is translated h units left. If $h > 0$, the parabola is translated h units right.
- The vertex of $y = (x - h)^2$ is the point $(h, 0)$.
- The equation of the axis of symmetry of $y = (x - h)^2$ is $x = h$.



- The value of k tells how far and in what direction the parabola is translated vertically. If $k < 0$, the parabola is translated k units down. If $k > 0$, the parabola is translated k units up.
- The vertex of $y = x^2 + k$ is the point $(0, k)$.
- The equation of the axis of symmetry of $y = x^2 + k$ is $x = 0$.



- The vertex of $y = (x - h)^2 + k$ is the point (h, k).
- The equation of the axis of symmetry of $y = (x - h)^2 + k$ is $x = h$.



FURTHER Your Understanding

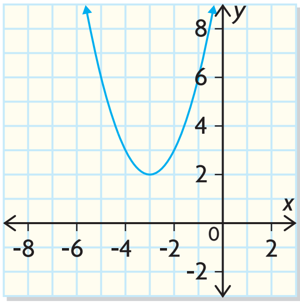
1. The following transformations are applied to a parabola with the equation $y = x^2$. Determine the values of h and k, and write the equation in the form $y = (x - h)^2 + k$.

a) The parabola moves 3 units right.
b) The parabola moves 4 units down.
c) The parabola moves 2 units left.
d) The parabola moves 5 units up.
e) The parabola moves 7 units down and 6 units left.
f) The parabola moves 2 units right and 5 units up.

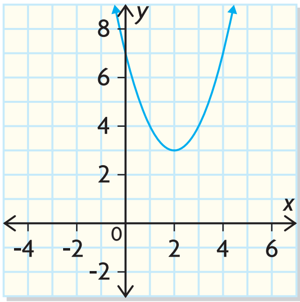
2. Match each equation with the correct graph.

a) $y = (x - 2)^2 + 3$
b) $y = (x + 2)^2 - 3$
c) $y = (x + 3)^2 - 2$
d) $y = (x - 3)^2 + 2$

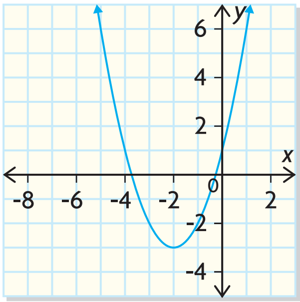
i)



iii)



v)



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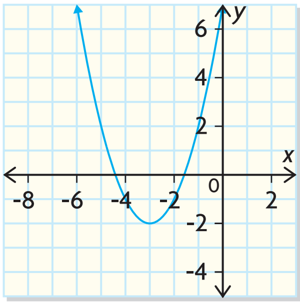

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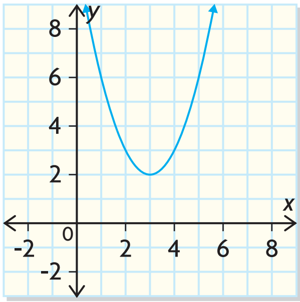

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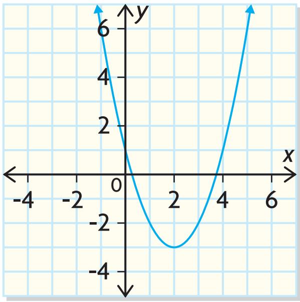


3. Sketch the graph of each relation by hand. Start with the graph of $y = x^2$, and apply the appropriate transformations.

a) $y = x^2 - 4$
b) $y = (x - 3)^2$
c) $y = x^2 + 2$
d) $y = (x + 5)^2$
e) $y = (x + 1)^2 - 2$
f) $y = (x - 5)^2 + 3$

4. Describe the transformations that are applied to the graph of $y = x^2$ to obtain the graph of each quadratic relation.

a) $y = x^2 + 5$
b) $y = (x - 3)^2$
c) $y = -3x^2$
d) $y = (x + 7)^2$
e) $y = \frac{1}{2}x^2$
f) $y = (x + 6)^2 + 12$

5. State the vertex and the axis of symmetry of each parabola in question 4.

5.3

Graphing Quadratics in Vertex Form

GOAL

Graph a quadratic relation in the form $y = a(x - h)^2 + k$ by using transformations.

YOU WILL NEED

- grid paper
- ruler

LEARN ABOUT the Math

Srinithi and Kevin are trying to sketch the graph of the quadratic relation $y = 2(x - 3)^2 - 8$ by hand. They know that they need to apply a series of transformations to the graph of $y = x^2$.

? How do you apply transformations to the quadratic relation $y = x^2$ to sketch the graph of $y = 2(x - 3)^2 - 8$?

EXAMPLE 1 Selecting a transformation strategy to graph a quadratic relation

Use transformations to sketch the graph of $y = 2(x - 3)^2 - 8$.

Srinithi's Solution: Applying a horizontal translation first

$y = x^2$

x	−2	−1	0	1	2
y	4	1	0	1	4

I began by graphing $y = x^2$ using five key points. The quadratic relation $y = 2(x - 3)^2 - 8$ is expressed in vertex form.

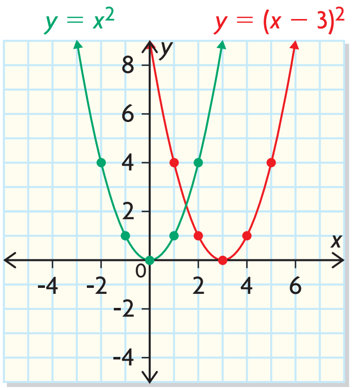
vertex form

a quadratic relation of the form $y = a(x - h)^2 + k$, where the vertex is (h, k)

$y = (x - 3)^2$

x	1	2	3	4	5
y	4	1	0	1	4

Since $h = 3$, I added 3 to the x-coordinate of each point on $y = x^2$. This means that the vertex is (3, 0).

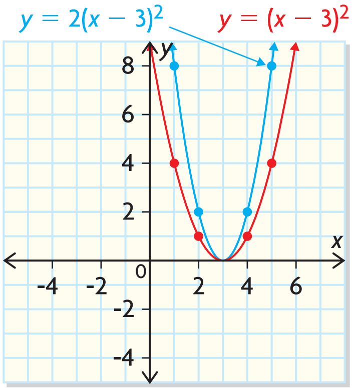


The equation of the new **red** graph is $y = (x - 3)^2$. To draw it, I translated the green parabola 3 units right.

$y = 2(x - 3)^2$

x	1	2	3	4	5
y	8	2	0	2	8

Since $a = 2$, I multiplied all the y-coordinates of the points on the red graph by 2. The vertex stays at (3, 0). The equation of this graph is $y = 2(x - 3)^2$.

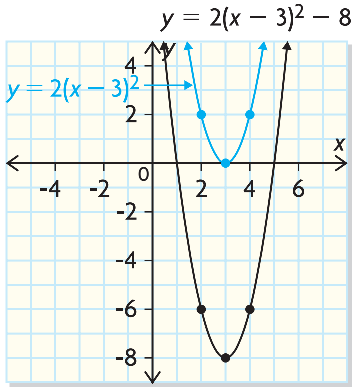


To draw this new **blue** graph, I applied a vertical stretch by a factor of 2 to the red graph. The blue graph looks correct because the graph with the greater *a* value should be narrower than the other graph.

$y = 2(x - 3)^2 - 8$

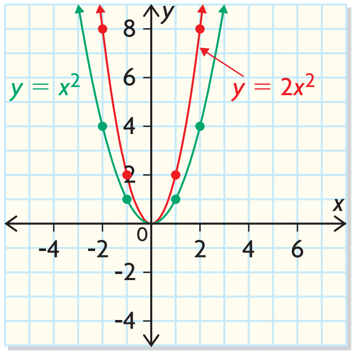
x	1	2	3	4	5
y	0	-6	-8	-6	0

I knew that $k = -8$. I subtracted 8 from the y-coordinate of each point on the blue graph. The vertex is now $(3, -8)$. The equation of the graph is $y = 2(x - 3)^2 - 8$.



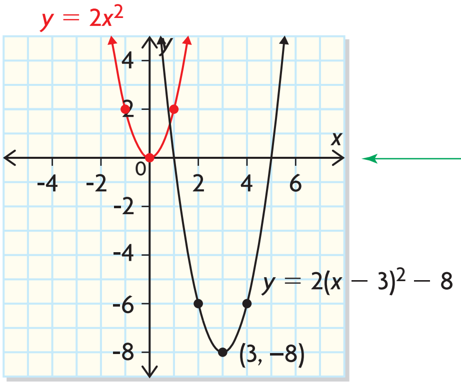
Since $k < 0$, I knew that I had to translate the blue graph 8 units down to get the final **black** graph.

Kevin's Solution: Applying a vertical stretch first



Since $a = 2$, I decided to stretch the graph of $y = x^2$ vertically by a factor of 2. I multiplied the y-coordinate of each point on the graph of $y = x^2$ by 2.

The equation of the resulting **red** graph is $y = 2x^2$.



I applied both translations in one step. Adding 3 to the *x*-coordinate and subtracting 8 from the *y*-coordinate from each point on the red graph causes the red graph to move 3 units right and 8 units down.

The equation of the resulting **black** graph is $y = 2(x - 3)^2 - 8$.

Reflecting

A. Why was it not necessary for Kevin to use two steps for the translations? In other words, why did he not have to shift the graph to the right in one step, and then down in another step?

B. What are the advantages and disadvantages of each solution?

C. How can thinking about the order of operations applied to the coordinates of points on the graph of $y = x^2$ help you apply transformations to draw a new graph?

APPLY the Math

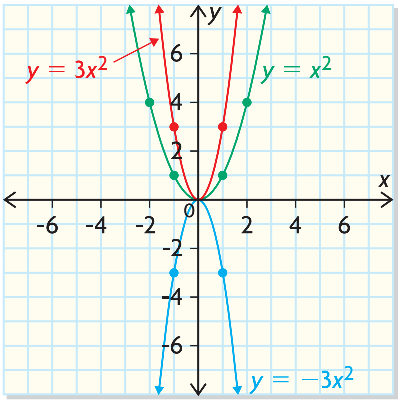
EXAMPLE 2 Reasoning about sketching the graph of a quadratic relation

Sketch the graph of $y = -3(x + 5)^2 + 1$, and explain your reasoning.

Winnie's Solution: Connecting a sequence of transformations to the equation

Applying a vertical stretch of factor 3 and a reflection in the *x*-axis gives the graph of $y = -3x^2$.

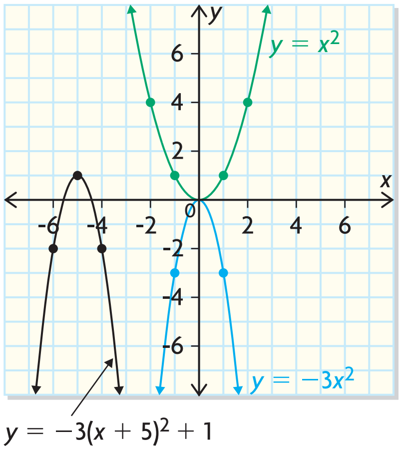
In the quadratic relation $y = -3(x + 5)^2 + 1$, the value of a is -3. This represents a vertical stretch by a factor of 3 and a reflection in the *x*-axis.



I noticed that I can combine the stretch and reflection into a single step by multiplying each *y*-coordinate of points on $y = x^2$ by -3.

In the equation, $h = -5$ and $k = 1$. Therefore, the vertex is at $(-5, 1)$. I translated the blue graph 5 units left and 1 unit up.

I determined that the vertex is $(-5, 1)$. Then I shifted all the points on the graph of $y = -3x^2$ so that they were 5 units left and 1 unit up.



I drew a smooth curve through the new points to sketch the graph.

Beth's Solution: Connecting the properties of a parabola to the equation

Based on the equation $y = -3(x + 5)^2 + 1$, the parabola has these properties:

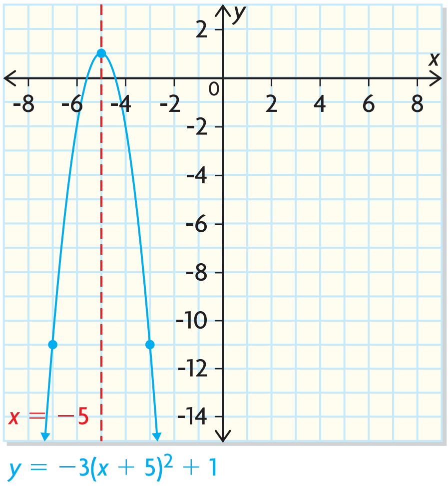
- Since $a < 0$, the parabola opens downward.
- The vertex of the parabola is $(-5, 1)$.
- The equation of the axis of symmetry is $x = -5$.

Since the equation was given in vertex form, I listed the properties of the parabola that I could determine from the equation.

$$y = -3(-3 + 5)^2 + 1$$
$$y = -3(2)^2 + 1$$
$$y = -12 + 1$$
$$y = -11$$

To determine another point on the parabola, I let $x = -3$.

Therefore, $(-3, -11)$ is a point on the parabola.

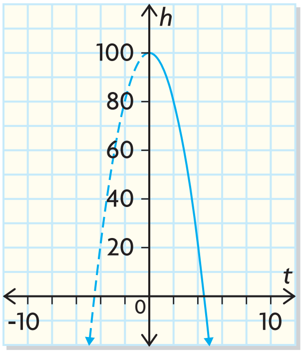


I plotted the vertex and the point I had determined, $(-3, -11)$. Then I drew the axis of symmetry. I used symmetry to determine the point directly across from $(-3, -11)$. This point is $(-7, -11)$.

I plotted the points and joined them with a smooth curve.

EXAMPLE 3 Reasoning about the effects of transformations on a quadratic relation

For a high school charity event, the principal pays to drop a watermelon from a height of 100 m. The height, h, in metres, of the watermelon after t seconds is $h = -0.5gt^2 + k$, where g is the acceleration due to gravity and k is the height from which the watermelon is dropped. On Earth, $g = 9.8$ m/s^2.



$h = -4.9t^2 + 100, t \geq 0$

a) The clock that times the fall of the watermelon runs for 3 s before the principal releases the watermelon. How does this change the graph shown? Determine the equation of the new relation.

b) On Mars, $g = 3.7$ m/s^2. Suppose that an astronaut dropped a watermelon from a height of 100 m on Mars. Determine the equation for the height of the watermelon on Mars. How does the graph for Mars compare with the graph for Earth for part a)?

c) The principal drops another watermelon from a height of 50 m on Earth. How does the graph for part a) change? How does the relation change?

d) Repeat part c) for an astronaut on Mars.

Nadia's Solution

a) The equation of the original relation is

$h = -0.5(9.8)t^2 + 100$

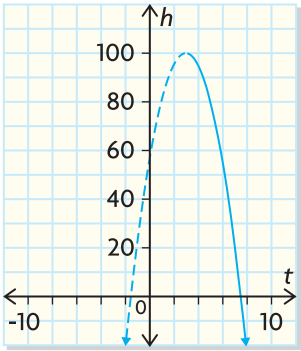
$h = -4.9t^2 + 100$, where $t \geq 0$

The original graph is a parabola that opens downward, with vertex $(0, k) = (0, 100)$. I wrote and simplified the original relation. Only the right branch of the parabola makes sense in this situation since time can't be negative.

The parabola is translated 3 units right. The equation of the new relation is

$h = -4.9(t - 3)^2 + 100$, where $t \geq 3$.

I subtracted 3 from the t-coordinate to determine the new relation. Since the watermelon is not falling before 3 s, the relation only holds for $t \geq 3$.

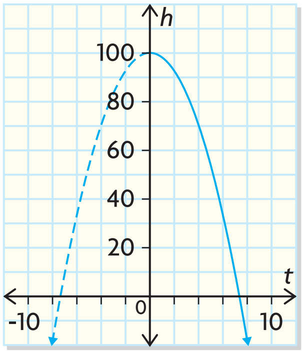


$h = -4.9(t - 3)^2 + 100, t \geq 3$

If the clock runs for 3 s before the watermelon is dropped, then the watermelon will be at its highest point at 3 s. So, the vertex of the new parabola is (3, 100), which is a shift of the original parabola 3 units right.

b) The equation of the relation on Mars is
$h = -0.5(3.7)t^2 + 100$
$h = -1.85t^2 + 100$, where $t \geq 0$
The graph for Mars is wider near the vertex.

I used the value of g on Mars, $g = 3.7\text{m/s}^2$, instead of $g = 9.8\text{ m/s}^2$.



$h = -1.85t^2 + 100, t \geq 0$

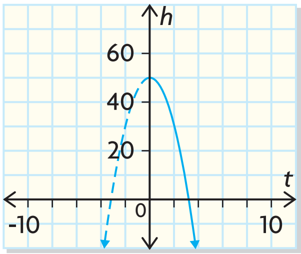
A lesser (negative) a-value means that the parabola is wider.

The t-intercept is farther from the origin, so the watermelon would take longer to hit the ground on Mars compared to Earth.

c) The equation of the new relation is
$h = -4.9t^2 + 50$, where $t \geq 0$.

In the relation, k changes from 100 to 50.

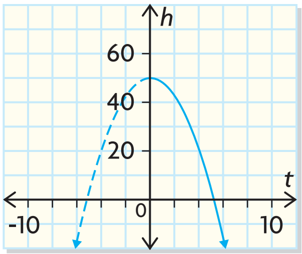
The new graph has the same shape but is translated 50 units down.



$h = -4.9t^2 + 50, t \geq 0$

The new vertex is half the distance above the origin, at (0, 50) instead of (0, 100). This is a shift of 50 units down.

d) The new graph for Mars is wider than the original graph and is translated 50 units down.



$h = -1.85t^2 + 50, t \geq 0$

The new graph for Mars is wider than the original graph, like the graph for part b). It is translated down, like the graph for part c).

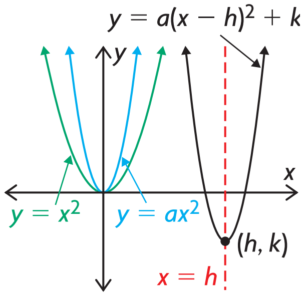
In Summary

Key Idea

- Compared with the graph of $y = x^2$, the graph of $y = a(x - h)^2 + k$ is a parabola that has been stretched or compressed vertically by a factor of a, translated horizontally by h, and translated vertically by k. As well, if $a < 0$, the parabola is reflected in the x-axis.

Need to Know

- The vertex of $y = a(x - h)^2 + k$ has the coordinates (h, k). The equation of the axis of symmetry of $y = a(x - h)^2 + k$ is $x = h$.
- When sketching the graph of $y = a(x - h)^2 + k$ as a transformation of the graph of $y = x^2$, follow the order of operations for the arithmetic operations to be performed on the coordinates of each point. Apply vertical stretches/compressions and reflections, which involve multiplication, before translations, which involve addition or subtraction.



CHECK *Your Understanding*

1. Describe the transformations you would apply to the graph of $y = x^2$, in the order you would apply them, to obtain the graph of each quadratic relation.

a) $y = x^2 - 3$

b) $y = (x + 5)^2$

c) $y = -\frac{1}{2}x^2$

d) $y = 4(x + 2)^2 - 16$

2. For each quadratic relation in question 1, identify

i) the direction in which the parabola opens

ii) the coordinates of the vertex

iii) the equation of the axis of symmetry

3. Sketch the graph of each quadratic relation. Start with a sketch of $y = x^2$, and then apply the appropriate transformations in the correct order.

a) $y = (x + 5)^2 - 4$

b) $y = -0.5x^2 + 8$

c) $y = 2(x - 3)^2$

d) $y = \frac{1}{2}(x - 4)^2 - 2$

PRACTISING

4. What transformations would you apply to the graph of $y = x^2$ to create the graph of each relation? List the transformations in the order you would apply them.

a) $y = -x^2 + 9$

b) $y = (x - 3)^2$

c) $y = (x + 2)^2 - 1$

d) $y = -x^2 - 6$

e) $y = -2(x - 4)^2 + 16$

f) $y = \frac{1}{2}(x + 6)^2 + 12$

g) $y = -\frac{1}{2}(x + 4)^2 - 7$

h) $y = 5(x - 4)^2 - 12$

5. Sketch a graph of each quadratic relation in question 4 on a separate grid. Use the properties of the parabola and additional points as necessary.

6. Match each equation with the correct graph.

a) $y = \frac{1}{2}(x - 2)^2 - 5$

b) $y = \frac{1}{2}(x - 4)^2 - 2$

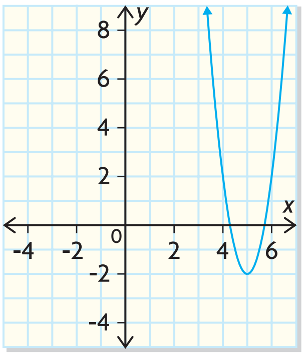
c) $y = -2(x + 2)^2 + 5$

d) $y = -2(x - 2)^2 - 5$

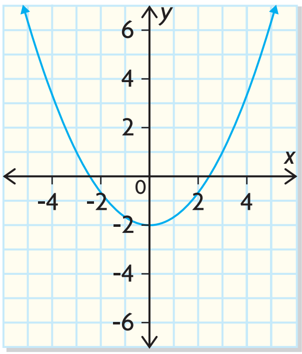
e) $y = 4(x - 5)^2 - 2$

f) $y = \frac{1}{3}x^2 - 2$

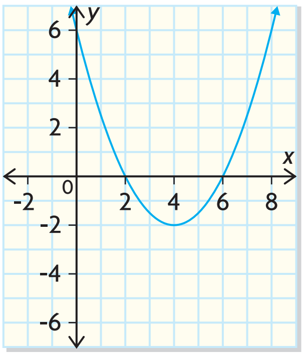
i)



iii)



v)



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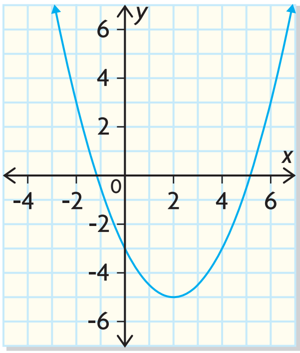

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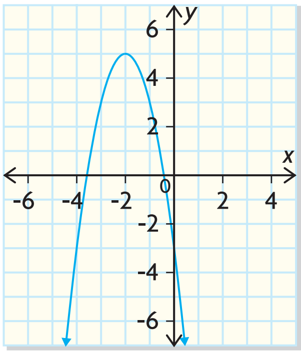
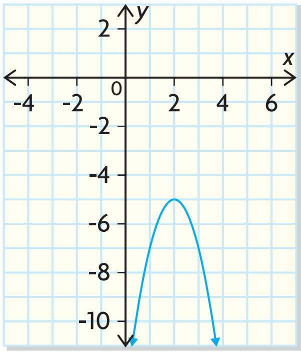


vi)



7. Sketch the graph of each quadratic relation by hand. Start with a sketch of $y = x^2$, and then apply the appropriate transformations in the correct order.

a) $y = -(x - 2)^2$

b) $y = \frac{1}{2}(x + 2)^2 - 8$

c) $y = -3(x - 1)^2 + 7$

d) $y = \frac{3}{4}x^2 - 5$

e) $y = \frac{1}{2}(x - 2)^2 - 5$

f) $y = -1.5(x + 3)^2 + 10$

8. Copy and complete the following table.

Quadratic Relation	Stretch/ Compression Factor	Reflection in the *x*-axis	Horizontal/ Vertical Translation	Vertex	Axis of Symmetry
	3	no	right 2, down 5	(2, −5)	$x = 2$
$y = 4(x + 2)^2 - 3$					
$y = -(x - 1)^2 + 4$					
$y = 0.8(x - 6)^2$					
$y = 2x^2 - 5$					

9. **C** Determine the equations of three different parabolas with a vertex at (−2, 3). Describe how the graphs of the parabolas are different from each other. Then sketch the graphs of the three relations on the same set of axes.

10. When an object with a parachute is released to fall freely, its height, h, in metres, after t seconds is modelled by $h = -0.5(g - r)t^2 + k$, where g is the acceleration due to gravity, r is the resistance offered by the parachute, and k is the height from which the object is dropped. On Earth, $g = 9.8$ m/s^2. The resistance offered by a single bed sheet is 0.6 m/s^2, by a car tarp is 2.1 m/s^2, and by a regular parachute is 8.9 m/s^2.

a) Describe how the graphs will differ for objects dropped from a height of 100 m using each of the three types of parachutes.

b) Is it possible to drop an object attached to the bed sheet and a similar object attached to a regular parachute and have them hit the ground at the same time? Describe how it would be possible and what the graphs of each might look like.

11. Write the equation of a parabola that matches each description.

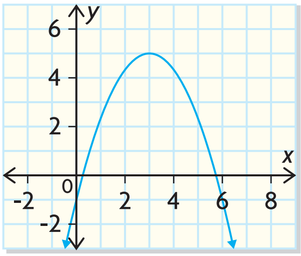
a) The graph of $y = x^2$ is reflected about the x-axis and then translated 5 units up.

b) The graph of $y = x^2$ is stretched vertically by a factor of 5 and then translated 2 units left.

c) The graph of $y = x^2$ is compressed vertically by a factor of $\frac{1}{5}$ and then translated 6 units down.

d) The graph of $y = x^2$ is reflected about the x-axis, stretched vertically by a factor of 6, translated 3 units right, and translated 4 units up.

12. **K** Sketch the graph of each parabola described in question 11 by applying the given sequence of transformations. Use a separate grid for each graph.



13. Which equation represents the graph shown at the left? Explain your reasoning.

a) $y = -\frac{2}{3}x^2 + 5$

b) $y = -(x - 3)^2 + 5$

c) $y = -\frac{2}{3}(x - 3)^2 + 5$

d) $y = \frac{2}{3}(x - 3)^2 + 5$

14. A sky diver jumped from an airplane. He used his watch to time the length of his jump. His height above the ground can be modelled by $h = -5(t - 4)^2 + 2500$, where h is his height above the ground in metres and t is the time in seconds from the time he started the timer.

a) How long did the sky diver delay his jump?

b) From what height did he jump?

Safety Connection

A helmet and goggles are important safety equipment for skydivers.

15. **A** A video tracking device recorded the height, h, in metres, of a baseball after it was hit. The data collected can be modelled by the relation $h = -5(t - 2)^2 + 21$, where t is the time in seconds after the ball was hit.

a) Sketch a graph that represents the height of the baseball.

b) What was the maximum height reached by the baseball?

c) When did the baseball reach its maximum height?

d) At what time(s) was the baseball at a height of 10 m?

e) Approximately when did the baseball hit the ground?

16. **T** When a graph of $y = x^2$ is transformed, the point (3, 9) moves to (8, 17). Describe three sets of transformations that could make this happen. For each set, give the equation of the new parabola.

17. Express the quadratic relation $y = 2(x - 4)(x + 10)$ in both standard form and vertex form.

18. Copy and complete the chart to show what you know about the quadratic relation $y = -2(x + 3)^2 + 4$.

Translation:	Reflection:
Stretch/ Compression:	Vertex:

$y = -2(x + 3)^2 + 4$

Extending

19. Determine one of the zeros of the quadratic relation

$$y = \left(x - \frac{k}{2}\right)^2 - \frac{(k - 2)^2}{4}.$$

5 Mid-Chapter Review

FREQUENTLY ASKED Questions

Q: How do you know whether the graph of $y = ax^2$ will have a wider or narrower shape near its vertex, compared with the graph of $y = x^2$?

A: The shape depends on the value of a in the equation. Each y-value is multiplied by a factor of a. When $a > 1$, the y-values increase. The parabola appears to be vertically stretched and becomes narrower near its vertex. When $0 < a < 1$, the y-values decrease. The parabola appears to be vertically compressed and becomes wider near its vertex.

Study Aid

- See Lesson 5.1, Examples 1 and 2.
- Try Mid-Chapter Review Questions 1 and 2.

Q: Why is the vertex form, $y = a(x - h)^2 + k$, useful for graphing quadratic relations?

A1: You can use the constants a, h, and k to determine how the graph of $y = x^2$ has been transformed.

- When $a > 1$, the parabola is vertically stretched and when $0 < a < 1$, the parabola is vertically compressed.
- When $a < 0$, the parabola is reflected in the x-axis.
- The parabola is translated to the right when $h > 0$ and to the left when $h < 0$. The parabola is translated up when $k > 0$ and down when $k < 0$.
- The coordinates of the vertex are (h, k).

A2: You can use the constants a, h, and k to determine key features of the parabola.

- When $a > 0$, the parabola opens upward. When $a < 0$, the parabola opens downward.
- The coordinates of the vertex are (h, k).
- The equation of the axis of symmetry is $x = h$.

You can use these properties, as well as the coordinates of a few other points, to draw an accurate sketch of any parabola.

Study Aid

- See Lesson 5.3, Examples 1 to 3.
- Try Mid-Chapter Review Questions 3, 4, and 6 to 8.

Q: When you use transformations to sketch a graph, why is the order in which you apply the transformations important?

A: When a graph is transformed, operations are performed on the coordinates of each point. Apply transformations in the same order you would apply calculations. Apply vertical stretches/compressions and reflections (multiplication) before translations (addition or subtraction).

Stretch → **Reflect** → **Translate**

Study Aid

- See Lesson 5.3, Examples 1 to 3.
- Try Mid-Chapter Review Question 5.

PRACTICE Questions

Lesson 5.1

1. Sketch the graph of each equation by correctly applying the required transformation(s) to points on the graph of $y = x^2$. Use a separate grid for each graph.

a) $y = 2x^2$

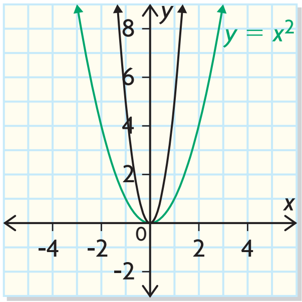
b) $y = -0.25x^2$

c) $y = -3x^2$

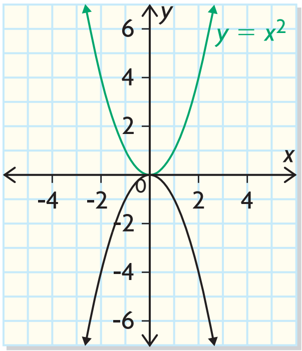
d) $y = \frac{2}{3}x^2$

2. Describe the transformation(s) that were applied to the graph of $y = x^2$ to obtain each **black** graph. Write the equation of the **black** graph.

a)



b)



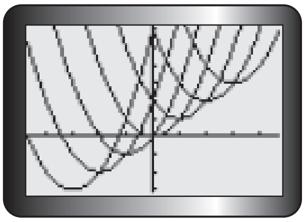
Lesson 5.2

3. Determine the values of h and k for each of the following transformations. Write the equation in the form $y = (x - h)^2 + k$. Sketch the graph.

a) The parabola moves 3 units down and 2 units right.

b) The parabola moves 4 units left and 6 units up.

4. These parabolas were entered as equations of the form $y = (x - h)^2 + k$. For each tick mark, the scale on both axes is 1. Determine as many of the equations as you can.



Lesson 5.3

5. Describe the sequence of transformations that you would apply to the graph of $y = x^2$ to sketch each quadratic relation.

a) $y = -3(x - 1)^2$

b) $y = \frac{1}{2}(x + 3)^2 - 8$

c) $y = 4(x - 2)^2 - 5$

d) $y = \frac{2}{3}x^2 - 1$

6. Sketch a graph of each quadratic relation in question 5 on a separate grid. Use the properties of the parabola and some additional points.

7. For each quadratic relation,

i) state the stretch/compression factor and the horizontal/vertical translations

ii) determine whether the graph is reflected in the x-axis

iii) state the vertex and the equation of the axis of symmetry

iv) sketch the graph by applying transformations to the graph of $y = x^2$

a) $y = (x - 2)^2 + 1$

b) $y = -\frac{1}{2}(x + 4)^2$

c) $y = 2(x + 1)^2 - 8$

d) $y = -0.25x^2 + 5$

8. A parabola lies in only two quadrants. What does this tell you about the values of a, h, and k? Explain your thinking, and provide the equation of a parabola as an example.

5.4

Quadratic Models Using Vertex Form

GOAL

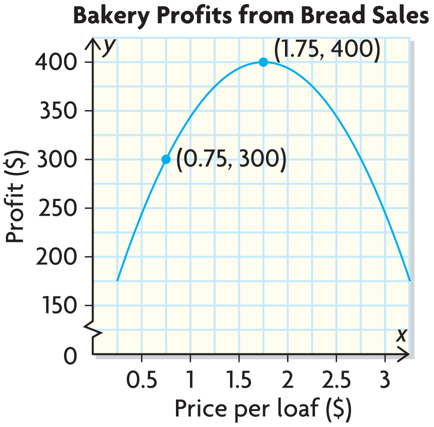
Write the equation of the graph of a quadratic relation in vertex form.

YOU WILL NEED

- grid paper
- ruler
- graphing calculator
- spreadsheet program (optional)

LEARN ABOUT the Math

The Best Bread Bakery wants to determine its daily profit from bread sales. This graph shows the data gathered by the company.



❓ What equation represents the relationship between the price of bread and the daily profit from bread sales?

EXAMPLE 1 | **Connecting** a parabola to the vertex form of its equation

Determine the equation of this quadratic relation from its graph.

Sabrina's Solution

$$y = a(x - h)^2 + k$$

Since the graph is a parabola and the coordinates of the vertex are given, I decided to use vertex form.

$$y = a(x - 1.75)^2 + 400$$

$$300 = a(0.75 - 1.75)^2 + 400$$

$$300 = a(-1)^2 + 400$$

$$300 = a + 400$$

$$-100 = a$$

The equation that represents the relationship is $y = -100(x - 1.75)^2 + 400$.

Since (1.75, 400) is the vertex, $h = 1.75$ and $k = 400$. I substituted these values into the equation.

To determine the value of a, I chose the point (0.75, 300) on the graph. I substituted these coordinates for x and y in the equation.

I followed the order of operations and solved for the value of a.

Reflecting

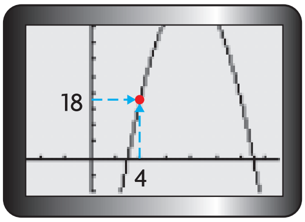
A. What information do you need from the graph of a quadratic relation to determine the equation of the relation in vertex form?

B. You have used the standard, factored, and vertex forms of a quadratic relation. Which form do you think is most useful for determining the equation of a parabola from its graph? Explain why.

APPLY the Math

EXAMPLE 2 **Connecting** information about a parabola to its equation

The graph of $y = x^2$ was stretched by a factor of 2 and reflected in the x-axis. The graph was then translated to a position where its vertex is not visible in the viewing window of a graphing calculator. Determine the quadratic relation in vertex form from the partial graph displayed in the screen shot. For each tick mark, the scale on the y-axis is 5, and the scale on the x-axis is 2.



Terri's Solution

$$a = -2$$
$$y = -2(x - h)^2 + k$$

The graph was stretched by a factor of 2 and reflected in the x-axis.

I substituted the value of a into the vertex form of the quadratic relation.

The zeros of the graph are 3 and 13.

$$h = \frac{3 + 13}{2}$$
$$h = 8$$

I determined the mean of the two zeros to calculate the value of h. The vertex lies on the axis of symmetry, which is halfway between the zeros of the graph.

$$18 = -2(4 - 8)^2 + k$$
$$18 = -2(16) + k$$
$$18 = -32 + k$$
$$50 = k$$

I saw that (4, 18) is a point on the graph. By substituting these coordinates, as well as the value I determined for h, I was able to solve for k.

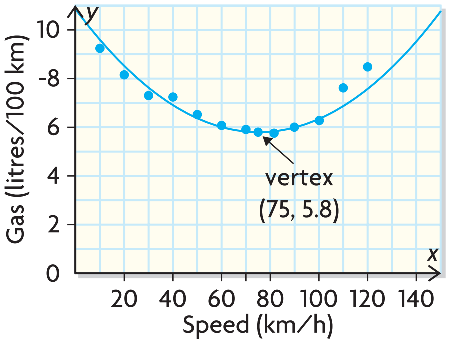
The equation of the graph is $y = -2(x - 8)^2 + 50$.

EXAMPLE 3 **Selecting a strategy** to determine a quadratic model

The amount of gasoline that a car consumes depends on its speed. A group of students decided to research the relationship between speed and fuel consumption for a particular car. They collected the data in the table. Determine an equation that models the relationship between speed and fuel consumption.

Speed (km/h)	10	20	30	40	50	60	70	80	90	100	110	120
Gas Consumed (litres/100 km)	9.2	8.1	7.4	7.2	6.4	6.1	5.9	5.8	6.0	6.3	7.5	8.4

Eric's Solution: Representing a relation with a scatter plot and determining the equation algebraically



I constructed a scatter plot to display the data and drew a curve of good fit. Since the curve looked parabolic and I knew that I could estimate the coordinates of the vertex. I estimated the coordinates of the vertex to be about (75, 5.8).

$$y = a(x - h)^2 + k$$
$$y = a(x - 75)^2 + 5.8$$

I decided to use the vertex form of the equation. I substituted the estimated values (75, 5.8) into the general equation.

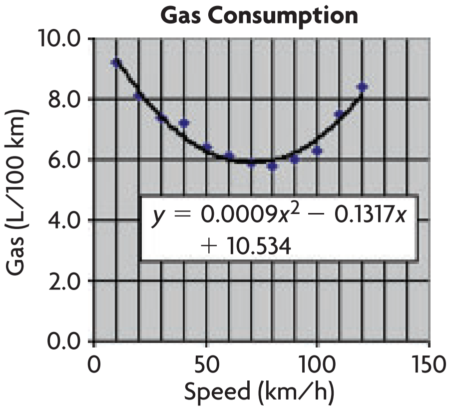
$$6.0 = a(90 - 75)^2 + 5.8$$

From the table, I knew that the point (90, 6.0) is close to the curve. I substituted the coordinates of this point for *x* and *y* to determine *a*.

$$6.0 = a(15)^2 + 5.8$$
$$6.0 = 225a + 5.8$$
$$0.2 = 225a$$
$$0.0009 \doteq a$$

I solved for *a*.

The equation that models the data is $y = 0.0009(x - 75)^2 + 5.8$.



I checked my equation using a spreadsheet. I entered the data from the table. I used column A for the *Speed* values and column B for the *Gas* values. I created a graph, added a trend line using **quadratic regression** of order 2, and chose the option to display the equation on the graph.

Tech | Support

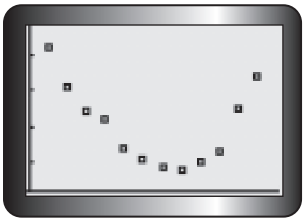
For help creating a scatter plot and performing a regression analysis using a spreadsheet, see Appendix B-35.

$$y = 0.0009(x - 75)^2 + 5.8$$
$$y = 0.0009(x^2 - 150x + 5625) + 5.8$$
$$y = 0.0009x^2 - 0.135x + 5.0625 + 5.8$$
$$y = 0.0009x^2 - 0.135x + 10.8625$$

The spreadsheet equation was in standard form, but my equation was in vertex form. To compare the two equations, I expanded my equation.

The two equations are very close, so they are both good quadratic models for this set of data.

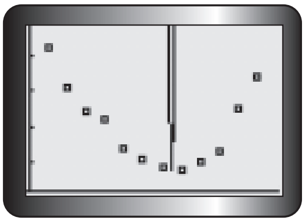
Gillian's Solution: Selecting a graphing calculator and an informal curve-fitting process



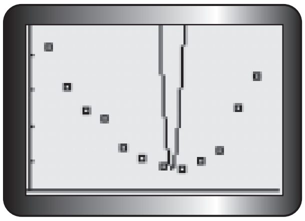
I entered the data into L1 and L2 in the data editor of a graphing calculator and created a scatter plot.

$$y = a(x - 75)^2 + 5.8$$

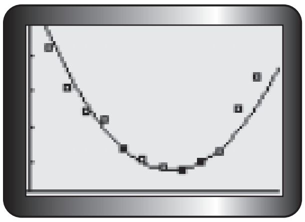
The points had a parabolic pattern, so I estimated the coordinates of the vertex to be about (75, 5.8). I substituted these coordinates into the general equation.



Since the parabola opens upward, I knew that $a > 0$. I used $a = 1$ and entered the equation $y = 1(x - 75)^2 + 5.8$ into Y1 of the equation editor. Then I graphed the equation. The location of the vertex looked good, but the parabola wasn't wide enough.



I decreased the value of a to $a = 0.1$, but the parabola still wasn't wide enough.

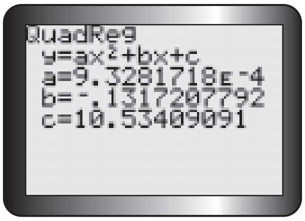


I decreased the value of a several more times until I got a good fit. I found that $a = 0.0009$ worked fairly well.

An equation that models the relationship between speed and fuel consumption is $y = 0.0009(x - 75)^2 + 5.8$.

Tech | ***Support***

For help creating a scatter plot using a TI-83/84 graphing calculator, see Appendix B-10. If you are using a TI-*n*spire, see Appendix B-46.



```
QuadReg
 y=ax²+bx+c
 a=9.3281718E-4
 b=-.1317207792
 c=10.53409091
```

I checked my equation by comparing it with the equation produced by quadratic regression on the graphing calculator. To do this, I had to expand my equation.

$$y = 0.0009(x - 75)^2 + 5.8$$
$$y = 0.0009(x^2 - 150x + 5625) + 5.8$$
$$y = 0.0009x^2 - 0.135x + 5.0625 + 5.8$$
$$y = 0.0009x^2 - 0.135x + 10.8625$$

The two equations are close, so they are both good models for this set of data.

Tech *Support*

For help performing a quadratic regression analysis using a TI-83/84 graphing calculator, see Appendix B-10. If you are using a TI-*n*spire, see Appendix B-46.

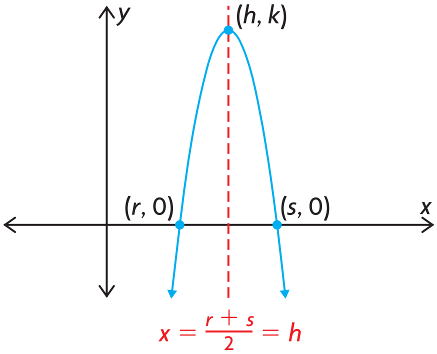
In Summary

Key Idea

- If you know the coordinates of the vertex (h, k) and one other point on a parabola, you can determine the equation of the relation using $y = a(x - h)^2 + k$.

Need to Know

- To determine the value of a, substitute the coordinates of a point on the graph into the general equation and solve for a:
  - If $(h, k) =$ (🟩, 🟦), then $y = a(x -$ 🟩$)^2 +$ 🟦.
  - If a point on the graph has coordinates $x =$ 🟪 and $y =$ 🟥, then, by substitution, 🟥 $= a($🟪 $-$ 🟩$)^2 +$ 🟦.
  - Since a is the only remaining unknown, its value can be determined by solving the equation.
- The vertex form of an equation can be determined using the zeros of the graph. The axis of symmetry is $x = h$, where h is the mean of the zeros.
- You can convert a quadratic equation from vertex form to standard form by expanding and then collecting like terms.



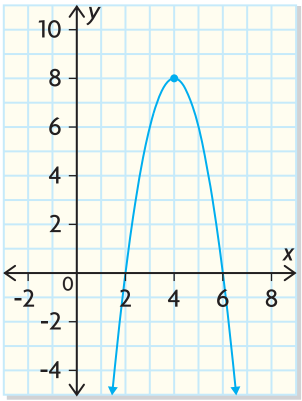
CHECK Your Understanding

1. Match each equation with the correct graph.

a) $y = 2x^2 - 8$ **c)** $y = -2(x - 4)^2 + 8$

b) $y = (x + 3)^2$ **d)** $y = (x - 3)^2 - 8$

i)



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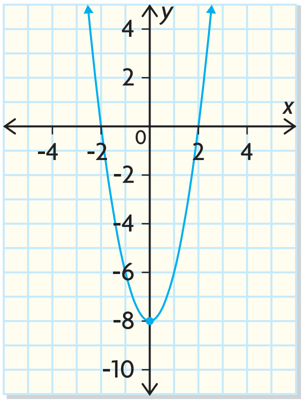

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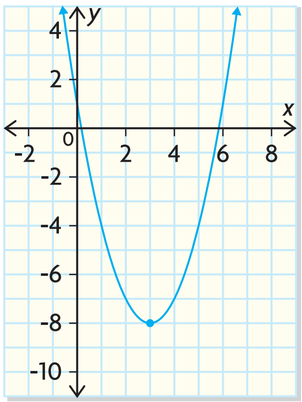

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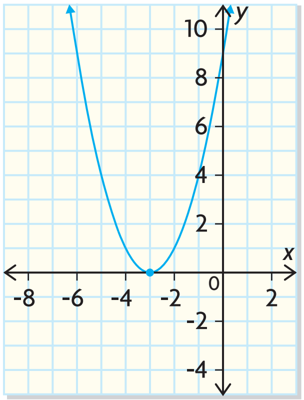


2. The vertex of a quadratic relation is (4, −12).

a) Write an equation to describe all parabolas with this vertex.

b) A parabola with the given vertex passes through point (13, 15). Determine the value of *a* for this parabola.

c) Write the equation of the relation for part b).

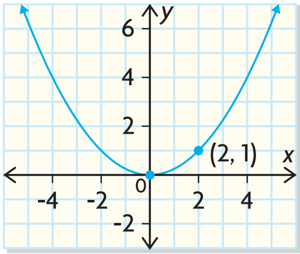
d) State the transformations that must be applied to $y = x^2$ to obtain the quadratic relation you wrote for part c).

e) Graph the quadratic relation you wrote for part c).

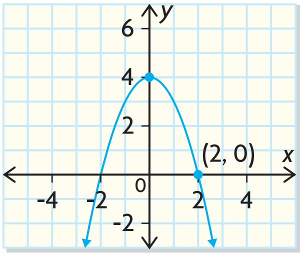
PRACTISING

3. Write the equation of each parabola in vertex form.

a)



c)



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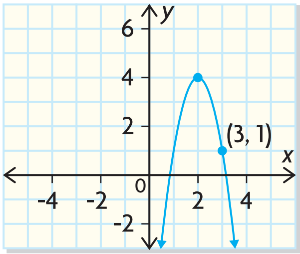

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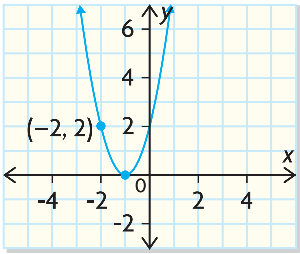

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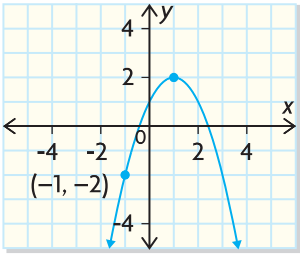

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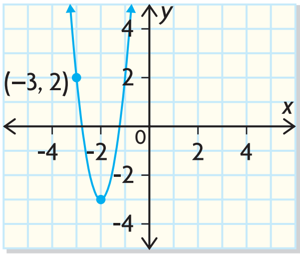


4. The following transformations are applied to the graph of $y = x^2$. Determine the equation of each new relation.

a) a vertical stretch by a factor of 4

b) a translation of 3 units left

c) a reflection in the x-axis, followed by a translation 2 units up

d) a vertical compression by a factor of $\frac{1}{2}$

e) a translation of 5 units right and 4 units down

f) a vertical stretch by a factor of 2, followed by a reflection in the x-axis and a translation 1 unit left

5. Write the equation of a parabola with each set of properties.

a) vertex at (0, 4), opens upward, the same shape as $y = x^2$

b) vertex at (5, 0), opens downward, the same shape as $y = x^2$

c) vertex at $(2, -3)$, opens upward, narrower than $y = x^2$

d) vertex at $(-3, 5)$, opens downward, wider than $y = x^2$

e) axis of symmetry $x = 4$, opens upward, two zeros, narrower than $y = x^2$

f) vertex at $(3, -4)$, no zeros, wider than $y = x^2$

6. Determine the equation of a quadratic relation in vertex form, given
K the following information.

a) vertex at $(-2, 3)$, passes through $(-4, 1)$

b) vertex at $(-1, -1)$, passes through $(0, 1)$

c) vertex at $(-2, -3)$, passes through $(-5, 6)$

d) vertex at $(-2, 5)$, passes through $(1, -4)$

7. Each table of values defines a parabola. Determine the equation of the axis of symmetry of the parabola, and write the equation in vertex form.

a)

x	y
2	−33
3	−13
4	−1
5	3
6	−1

b)

x	y
0	12
1	4
2	4
3	12
4	28

8. A child kicks a soccer ball so that it barely clears a 2 m fence. The soccer ball lands 3 m from the fence. Determine the equation, in vertex form, of a quadratic relation that models the path of the ball.

9. Data for DVD sales in Canada, over several years, are given in the table.

Year	2002	2003	2004	2005	2006
***x*, Years Since 2002**	0	1	2	3	4
DVDs Sold (1000s)	1446	3697	4573	4228	3702

a) Using graphing technology, create a scatter plot to display the data.
b) Estimate the vertex of the graph you created for part a). Then determine an equation in vertex form to model the data.
c) How many DVDs would you expect to be sold in 2010?
d) Check the accuracy of your model using quadratic regression.

10. **A** A school custodian finds a tennis ball on the roof of the school and throws it to the ground below. The table gives the height of the ball above the ground as it moves through the air.

Time (s)	0.0	0.5	1.0	1.5	2.0	2.5	3.0
Height (m)	5.00	11.25	15.00	16.25	15.00	11.25	5.00

a) Do the data appear to be linear or quadratic? Explain.
b) Create a scatter plot, and draw a quadratic curve of good fit.
c) Estimate the coordinates of the vertex.
d) Determine an algebraic relation in vertex form to model the data.
e) Use your model to predict the height of the ball at 2.75 s and 1.25 s.
f) How effective is your model for time values that are greater than 3.5 s? Explain.
g) Check the accuracy of your model using quadratic regression.

11. A chain of ice cream stores sells $840 of ice cream cones per day. Each ice cream cone costs $3.50. Market research shows the following trend in revenue as the price of an ice cream cone is reduced.

Price ($)	3.50	3.00	2.50	2.00	1.50	1.00	0.50
Revenue ($)	840	2520	3600	4080	3960	3240	1920

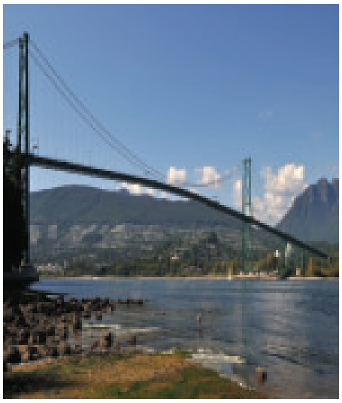
a) Create a scatter plot, and draw a quadratic curve of good fit.
b) Determine an equation in vertex form to model this relation.
c) Use your model to predict the revenue if the price of an ice cream cone is reduced to $2.25.
d) To maximize revenue, what should an ice cream cone cost?
e) Check the accuracy of your model using quadratic regression.

12. This table shows the number of imported cars that were sold in Newfoundland between 2003 and 2007.

Year	2003	2004	2005	2006	2007
Sales of Imported Cars (number sold)	3996	3906	3762	3788	4151

a) Create a scatter plot, and draw a quadratic curve of good fit.
b) Determine an algebraic equation in vertex form to model this relation.
c) Use your model to predict how many imported cars were sold in 2008.
d) What does your model predict for 2006? Is this prediction accurate? Explain.
e) Check the accuracy of your model using quadratic regression.

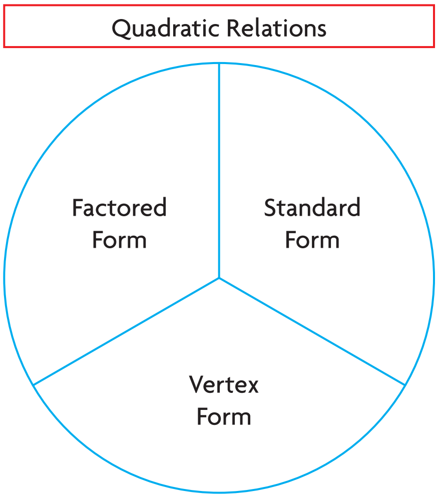
13. **T** The Lion's Gate Bridge in Vancouver, British Columbia, is a suspension bridge that spans a distance of 1516 m. Large cables are attached to the tops of the towers, 50 m above the road. The road is suspended from the large cables by many smaller vertical cables. The smallest vertical cable measures about 2 m. Use this information to determine a quadratic model for the large cables.



14. **C** A model rocket is launched from the ground. After 20 s, the rocket reaches a maximum height of 2000 m. It lands on the ground after 40 s. Explain how you could determine the equation of the relationship between the height of the rocket and time using two different strategies.

15. The owner of a small clothing company wants to create a mathematical model for the company's daily profit, p, in dollars, based on the selling price, d, in dollars, of the dresses made. The owner has noticed that the maximum daily profit the company has made is \$1600. This occurred when the dresses were sold for \$75 each. The owner also noticed that selling the dresses for \$50 resulted in a profit of \$1225. Using a quadratic relation to model this problem, create an equation for the company's daily profit.

16. Compare the three forms of the equation of a quadratic relation using this concept circle. Under what conditions would you use one form instead of the other forms when trying to connect a graph to its equation? Explain your thinking.



Extending

17. The following transformations are applied to a parabola with the equation $y = 2(x + 3)^2 - 1$. Determine the equation that will result after each transformation.

a) a translation 4 units right

b) a reflection in the x-axis

c) a reflection in the x-axis, followed by a translation 5 units down

d) a stretch by a factor of 6

e) a compression by a factor of $\frac{1}{4}$, followed by a reflection in the y-axis

18. The vertex of the parabola $y = 3x^2 + bx + c$ is at $(-1, 4)$. Determine the values of b and c.

19. Determine an algebraic expression for the solution, x, to the equation $0 = a(x - h)^2 + k$. Do not expand the equation.

5.5

Solving Problems Using Quadratic Relations

GOAL

Model and solve problems using the vertex form of a quadratic relation.

YOU WILL NEED

- grid paper
- ruler

LEARN ABOUT the Math

Smoke jumpers are firefighters who parachute into remote locations to suppress forest fires. They are often the first people to arrive at a fire. When smoke jumpers exit an airplane, they are in free fall until their parachutes open.

A quadratic relation can be used to determine the height, H, in metres, of a jumper t seconds after exiting an airplane. In this relation, $a = -0.5g$, where g is the acceleration due to gravity. On Earth, $g = 9.8\text{ m/s}^2$.

? If a jumper exits an airplane at a height of 554 m, how long will the jumper be in free fall before the parachute opens at 300 m?

Environment *Connection*

In a recent year, 3596 of the 7290 forest fires in Canada were caused by human activities such as careless smoking, campfires, use of welding equipment, or operation of a motor vehicle.

EXAMPLE 1 **Connecting** information from a problem to a quadratic model

a) Determine the quadratic relation that will model the height, H, of the smoke jumper at time t.

b) Determine the length of time that the jumper is in free fall.

Conor's Solution

a) $H = a(t - h)^2 + k$

I decided to use the vertex form of the quadratic relation because the problem contains information about the vertex.

$$H = a(t - 0)^2 + 554$$

The vertex is the point at which the jumper exited the plane. So the vertex has coordinates (0, 554). I substituted these coordinates into the general equation.

$$H = -0.5(9.8)(t - 0)^2 + 554$$

$H = -4.9(t - 0)^2 + 554$ is an equation in vertex form for the quadratic relation that models this situation.

Since $a = -0.5g$ and $g = 9.8\text{ m/s}^2$, I substituted these values into the vertex form of the equation.

$H = -4.9t^2 + 554$ is an equation in standard form for the quadratic relation that models this situation.

I noticed that the value of a is the same in both vertex form and standard form. This makes sense because the parabolas would not be congruent if they were different.

b)

$$\begin{aligned} 300 &= -4.9t^2 + 554 \\ -254 &= -4.9t^2 \\ \frac{-254}{-4.9} &= \frac{-4.9t^2}{-4.9} \\ 51.84 &= t^2 \\ \sqrt{51.84} &= t \end{aligned}$$

Because the parachute opened at 300 m, I substituted 300 for H. Then I solved for t.

$$7.2 \doteq t\text{, since } t > 0$$

The jumper is in free fall for about 7.2 s.

In this situation, time can't be negative. So, I didn't use the negative square root of 51.84.

Reflecting

A. Why was zero used for the t-coordinate of the vertex?

B. How would the equation change if the jumper hesitated for 2 s before exiting the airplane, after being given the command to jump?

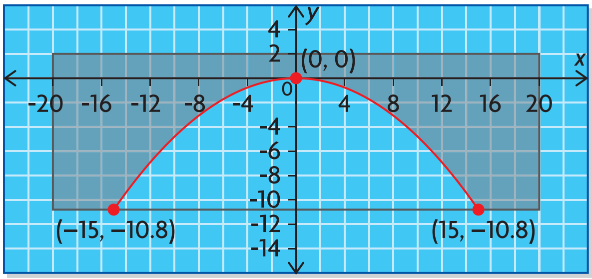
C. Why was the vertex form easier to use than either of the other two forms of a quadratic relation in this problem?

APPLY the Math

EXAMPLE 2 Solving a problem using a quadratic model

The underside of a concrete railway underpass forms a parabolic arch. The arch is 30.0 m wide at the base and 10.8 m high in the centre. The upper surface of the underpass is 40.0 m wide. The concrete is 2.0 m thick at the centre. Can a truck that is 5 m wide and 7.5 m tall get through this underpass?

Lisa's Solution



I started by drawing a diagram. I used a grid and marked the top of the arch as (0, 0). The upper surface of the underpass is 2 m above the top of the arch at the centre. The arch is 10.8 m high in the centre and 30 m wide at the base (or 15 m wide on each side). I marked the points (−15, −10.8) and (15, −10.8) and drew a parabola through these two points and the origin.

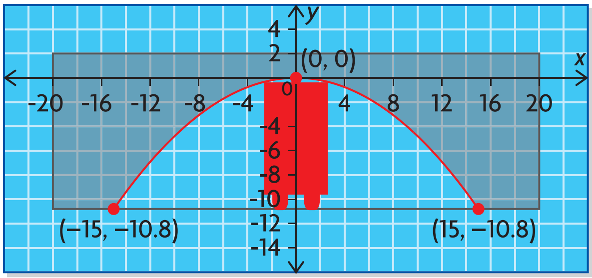
$$y = ax^2$$

The vertex of the parabola is at the origin, so I did not translate $y = ax^2$.

$$\begin{aligned} -10.8 &= a(15)^2 \\ -10.8 &= 225a \\ -0.048 &= a \end{aligned}$$

I determined the value of a by substituting the coordinates of a point on the graph, (15, −10.8), into this equation. Then I solved for a.

$y = -0.048x^2$ is the quadratic relation that models the arch of the railway underpass.



The truck has the best chance of getting through the underpass if it passes through the centre. Since the truck is 5 m wide, this means that the position of the right corner of the truck has an x-coordinate of 2.5. I substituted $x = 2.5$ into the equation to check the height of the underpass at this point.

$$\begin{aligned} y &= -0.048(2.5)^2 \\ y &= -0.048(6.25) \\ y &= -0.3 \end{aligned}$$

$$\begin{aligned} \text{Height at } (2.5, -0.3) &= 10.8 - 0.3 \\ &= 10.5 \end{aligned}$$

I determined the height from the ground at this point by subtracting 0.3 from 10.8.

The truck can get through. Since the truck is 7.5 m tall, there is 3 m of clearance.

The truck can get through the underpass, even if it is a little off the centre of the underpass.

EXAMPLE 3 Selecting a strategy to determine the vertex form

Write the quadratic relation $y = x^2 - 4x - 5$ in vertex form, and sketch the graph by hand.

Coral's Solution

$$y = x^2 - 4x - 5$$
$$y = (x + 1)(x - 5)$$

I rewrote the equation of the quadratic relation in factored form because I knew that I could determine the coordinates of the vertex from this form.

Zeros:

$0 = (x + 1)(x - 5)$

$x = -1$ and $x = 5$

The axis of symmetry is

$x = \dfrac{-1 + 5}{2}$ so $x = 2$.

I set $y = 0$ to determine the zeros. I used the zeros to determine the equation of the axis of symmetry.

$$y = (2)^2 - 4(2) - 5$$
$$y = 4 - 8 - 5$$
$$y = -9$$

I substituted $x = 2$ into the standard form of the equation to solve for y.

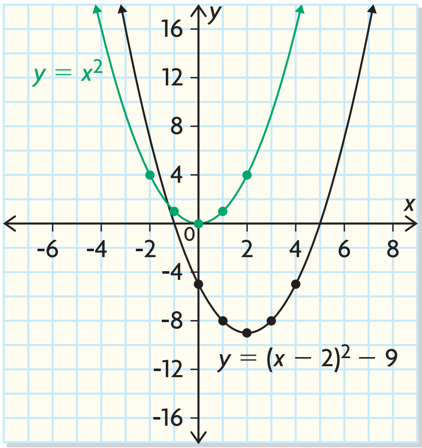
The vertex is at $(h, k) = (2, -9)$.

The coefficient of x^2 is $a = 1$.

I knew that the value of a must be the same in the standard, factored, and vertex forms. If it were different, the parabola would have different widths.

The relation is $y = (x - 2)^2 - 9$ in vertex form.

I substituted what I knew into the vertex form, $y = a(x - h)^2 + k$.



I sketched the graph of $y = x^2$ and translated each point 2 units right and 9 units down.

EXAMPLE 4 Representing a situation with a quadratic model

The Next Cup coffee shop sells a special blend of coffee for \$2.60 per mug. The shop sells about 200 mugs per day. Customer surveys show that for every \$0.05 decrease in the price, the shop will sell 10 more mugs per day.

a) Determine the maximum daily revenue from coffee sales and the price per mug for this revenue.

b) Write an equation in both standard form and vertex form to model this problem. Then sketch the graph.

Dave's Solution: Connecting the zeros of a parabola to the vertex form of the equation

a) Let x represent the number of \$0.05 decreases in price, where Revenue = (price)(mugs sold).

I defined a variable that connects the price per mug to the number of mugs sold.

$$r = (2.60 - 0.05x)(200 + 10x)$$

I used the information in the problem to write expressions for the price per mug and the number of mugs sold in terms of x. If I drop the price by \$0.05, x times, then the price per mug is $2.60 - 0.05x$ and the number of mugs sold is $200 + 10x$.

I used my expressions to write a relationship for daily revenue, r.

$$0 = (2.60 - 0.05x)(200 + 10x)$$

$$2.60 - 0.05x = 0, \text{ so } x = 52$$

or

$$200 + 10x = 0, \text{ so } x = -20$$

Since the equation is in factored form, the zeros of the equation can be calculated by letting $r = 0$ and solving for x.

$$x = \frac{52 + (-20)}{2}$$

$$x = 16$$

I used the zeros to determine the equation of the axis of symmetry.

$$r = [2.60 - 0.05(16)][200 + 10(16)]$$

$$r = (1.80)(360)$$

$$r = 648$$

The maximum daily revenue is \$648.

The maximum value occurs at the vertex. To calculate it, I substituted the x-value for the axis of symmetry into the revenue equation.

Price per mug for maximum revenue

$$= 2.60 - 0.05(16)$$
$$= 1.80$$

I substituted $x = 16$ into the expression for the price per mug.

The coffee shop should sell each mug of coffee for $1.80 to achieve a maximum daily revenue of $648.

b) In this relation, the maximum value is $r = 648$. It occurs when $x = 16$.

The vertex is $(16, 648) = (h, k)$.

$$r = a(x - 16)^2 + 648$$

The maximum value occurs at the vertex of a quadratic relation. To write the equation in vertex form, substitute the values of *h* and *k* into the vertex form of the general equation.

When $x = 0$,

$$r = (2.60)(200) = 520$$
$$520 = a(0 - 16)^2 + 648$$
$$520 = a(-16)^2 + 648$$
$$-128 = 256a$$
$$-0.5 = a$$

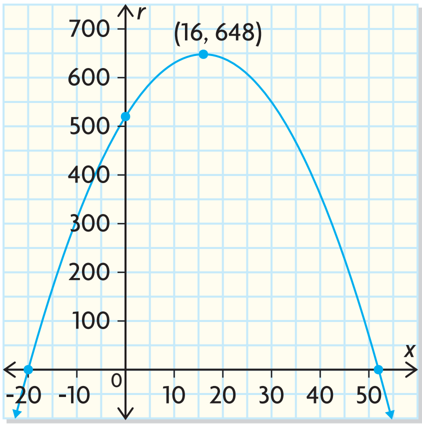
The equation in vertex form is $r = -0.5(x - 16)^2 + 648$.

Since the coffee shop sells 200 mugs of coffee when the price is $2.60 per mug, the point (0, 520) is on the graph. I substituted these coordinates into the equation and solved for *a*.

$$r = -0.5(x - 16)^2 + 648$$
$$r = -0.5(x^2 - 32x + 256) + 648$$
$$r = -0.5x^2 + 16x - 128 + 648$$
$$r = -0.5x^2 + 16x + 520$$

I expanded to get the equation in standard form.

The equation in standard form is $r = -0.5x^2 + 16x + 520$.

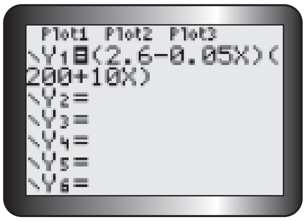


The vertex is at (16, 648). The zeros are at (52, 0) and (−20, 0).
The *y*-intercept is at (0, 520).

I used these points to sketch the graph of the relation.

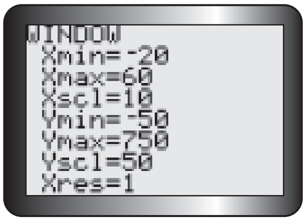
Toni's Solution: Selecting a graphing calculator to determine the quadratic model

a)



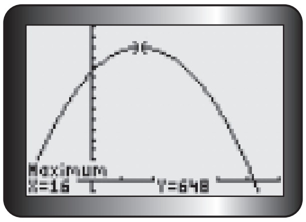
```
Plot1 Plot2 Plot3
\Y1■(2.6-0.05X)(
200+10X)
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
```

Once I created the revenue equation, I entered it into the equation editor as Y1.



```
WINDOW
 Xmin=-20
 Xmax=60
 Xscl=10
 Ymin=-50
 Ymax=750
 Yscl=50
 Xres=1
```

I graphed the revenue equation. I had to adjust the window settings until I could see the zeros and the vertex.



```
Maximum
X=16          Y=648
```

Since the vertex in this model represents the maximum value, I determined it using the maximum operation.

The vertex is at (16, 648).

The maximum daily revenue is \$648.

$$\text{Selling price} = 2.60 - 0.05(16)$$
$$= 1.80$$

Each mug of coffee should be sold for \$1.80 to maximize the daily revenue.

The maximum value is $y = 648$. This means that the maximum daily revenue is \$648. It occurs when $x = 16$.

b) The equation in standard form is

$$y = (2.60 - 0.05x)(200 + 10x)$$
$$y = -0.5x^2 + 16x + 520$$

Since the calculator has already produced the graph of the model, I only needed to determine the vertex form. I took the revenue equation and expanded it to get the equation in standard form.

$a = -0.5$

The vertex is at $(16, 648) = (h, k)$.

$$y = a(x - h)^2 + k$$
$$y = -0.5(x - 16)^2 + 648$$

This is the equation in vertex form.

Since the value of a is the same in all forms of a quadratic relation, I used it along with the coordinates of the vertex, and substituted to obtain the equation in vertex form.

Tech | ***Support***

For help determining the maximum value of a relation using a TI-83/84 graphing calculator, see Appendix B-9. If you are using a TI-*n*spire, see Appendix B-45.

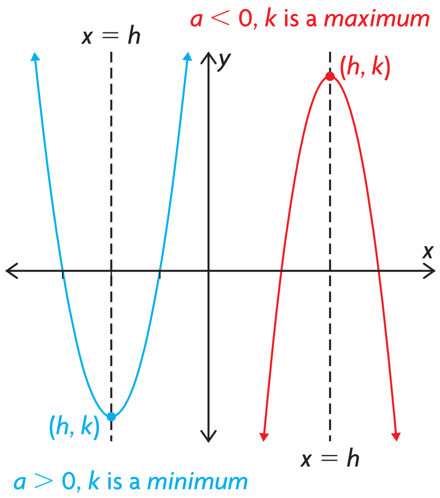
In Summary

Key Idea

- All quadratic relations can be expressed in vertex form and standard form. Quadratic relations that have zeros can also be expressed in factored form.
- For any parabola, the value of a is the same in all three forms of the equation of the quadratic relation.

Need to Know

- The y-coordinate of the vertex of a parabola represents the maximum or minimum value of the quadratic relation. The coordinates of the vertex are easily determined from the vertex form of the equation.
- If a situation can be modelled by a quadratic relation of the form $y = a(x - h)^2 + k$, the maximum or minimum value of y is k and it occurs when $x = h$.



- If $y = ax^2 + bx + c$ can be factored as a product of first-degree binomials and a constant, $y = a(x - r)(x - s)$, then this equation can be used to determine the vertex form of the quadratic relation as follows:
  - Use $x = \frac{r + s}{2}$ to determine the equation of the axis of symmetry. This gives you the value of h.
  - Substitute $x = \frac{r + s}{2}$ into $y = ax^2 + bx + c$ to determine the y-coordinate of the vertex. This gives you the value of k.
  - Substitute the values of a, h, and k into $y = a(x - h)^2 + k$.

CHECK Your Understanding

1. Use the given information to determine the equation of each quadratic relation in vertex form, $y = a(x - h)^2 + k$.

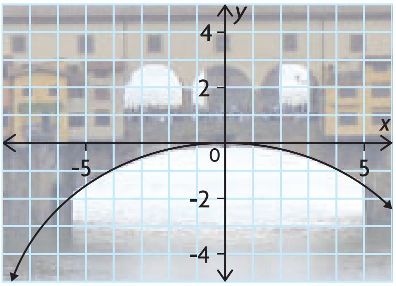
a) $a = 2$, vertex at $(0, 3)$
b) $a = -3$, vertex at $(2, 0)$
c) $a = -1$, vertex at $(3, -2)$
d) $a = 0.5$, vertex at $(-3.5, 18.3)$

2. Determine each maximum or minimum value in question 1.

3. The arch of the bridge in this photograph can be modelled by a parabola.

a) Determine an equation of the parabola.

b) On the upper part of the bridge, three congruent arches are visible in the first and second quadrants. What can you conclude about the value of a in the equations of the parabolas that model these arches? Explain.



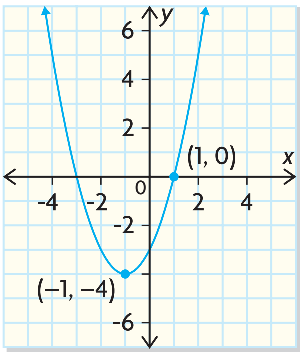
PRACTISING

4. Determine the equation of a quadratic relation in vertex form, given the following information.

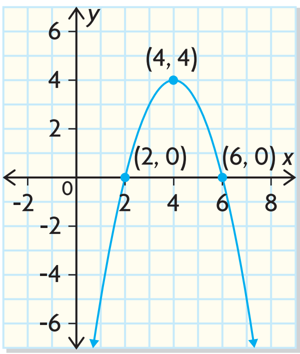
a) vertex at $(0, 3)$, passes through $(2, -5)$
b) vertex at $(2, 0)$, passes through $(5, 9)$
c) vertex at $(-3, 2)$, passes through $(-1, 14)$
d) vertex at $(5, -3)$, passes through $(1, -8)$

5. Determine the equation of each parabola in vertex form.

a)



c)



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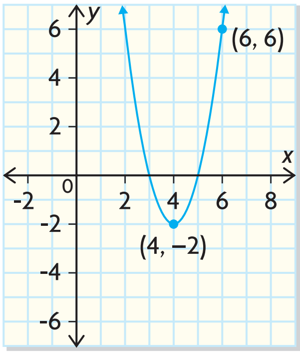
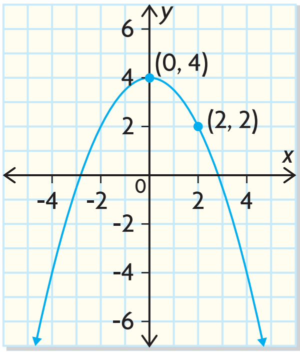


d)



6. Write each equation in question 5 in standard form and factored form.

7. A quadratic relation has zeros at –2 and 8, and a y-intercept of 8.
K Determine the equation of the relation in vertex form.

8. The quadratic relation $y = 2(x + 4)^2 - 7$ is translated 5 units right and 3 units down. What is the minimum value of the new relation? Write the equation of this relation in vertex form.

9. Express each equation in standard form and factored form.

a) $y = (x - 4)^2 - 1$

b) $y = 2(x + 1)^2 - 18$

c) $y = -(x + 5)^2 + 1$

d) $y = -3(x + 3)^2 + 75$

10. Express each equation in factored form and vertex form.

a) $y = 2x^2 - 12x$

b) $y = -2x^2 + 24x - 64$

c) $y = 2x^2 - x - 6$

d) $y = 4x^2 + 20x + 25$

11. A dance club has a \$5 cover charge and averages 300 customers on Friday nights. Over the past several months, the club has changed the cover price several times to see how this affects the number of customers. For every increase of \$0.50 in the cover charge, the number of customers decreases by 30. Use an algebraic model to determine the cover charge that maximizes revenue.

12. The graph of $y = -2(x + 5)^2 + 8$ is translated so that its new zeros are -4 and 2. Determine the translation that was applied to the original graph.

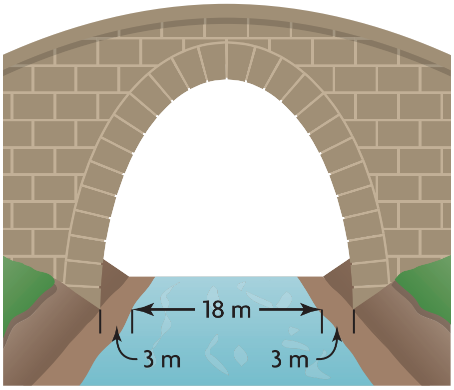
13. The average ticket price at a regular movie theatre (all ages) from 1995 to 1999 can be modelled by $C = 0.06t^2 - 0.27t + 5.36$, where C is the price in dollars and t is the number of years since 1995 ($t = 0$ for 1995, $t = 1$ for 1996, and so on).

a) When were ticket prices the lowest during this period?

b) What was the average ticket price in 1998?

c) What does the model predict the average ticket price will be in 2010?

d) Write the equation for the model in vertex form.



14. A bridge is going to be constructed over a river. The underside of the
A bridge will form a parabolic arch, as shown in the picture. The river is 18 m wide and the arch will be anchored on the ground, 3 m back from the riverbank on both sides. The maximum height of the arch must be between 22 m and 26 m above the surface of the river. Create two different equations to represent arches that satisfy these conditions. Then use graphing technology to graph your equations on the same grid.

15. A movie theatre can accommodate a maximum of 350 moviegoers per day. The theatre operators have been changing the admission price to find out how price affects ticket sales and profit. Currently, they charge \$11 a person and sell about 300 tickets per day. After reviewing their data, the theatre operators discovered that they could express the relation between profit, P, and the number of \$1 price increases, x, as $P = 20(15 - x)(11 + x)$.

a) Determine the vertex form of the profit equation.

b) What ticket price results in the maximum profit? What is the maximum profit? About how many tickets will be sold at this price?

16. The underside of a bridge forms a parabolic arch. The arch has a maximum height of 30 m and a width of 50 m. Can a sailboat pass under the bridge, 8 m from the axis of symmetry, if the top of its mast is 27 m above the water? Justify your solution.

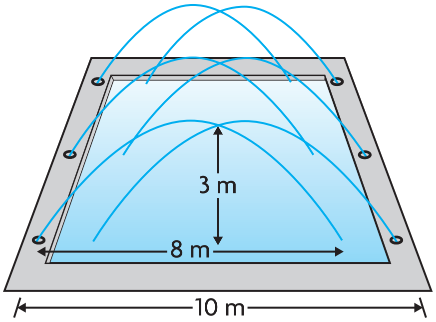
17. **T** A parabola has a y-intercept of -4 and passes through points $(-2, 8)$ and $(1, -1)$. Determine the vertex of the parabola.

18. **C** Serena claims that the standard form of a quadratic relation is best for solving problems where you need to determine the maximum or minimum value, and that the vertex form is best to use to determine a parabola's zeros. Do you agree or disagree? Explain.

Extending

19. The equation of a parabola is $y = a(x - 1)^2 + q$, and the points $(-1, -9)$ and $(1, 1)$ lie on the parabola. Determine the maximum value of y.

20. A rectangular swimming pool has a row of water fountains along each of its two longer sides. The two rows of fountains are 10 m apart. Each fountain sprays an identical parabolic-shaped stream of water a total horizontal distance of 8 m toward the opposite side. Looking from one end of the pool, the streams of water from the two sides cross each other in the middle of the pool at a height of 3 m.

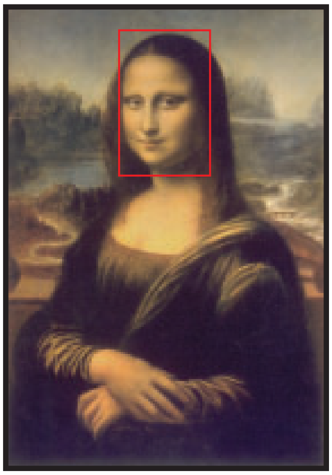


a) Determine an equation that represents a stream of water from the left side and another equation that represents a stream of water from the right side. Graph both equations on the same set of axes.

b) Determine the maximum height of the water.

YOU WILL NEED

- graphing calculator



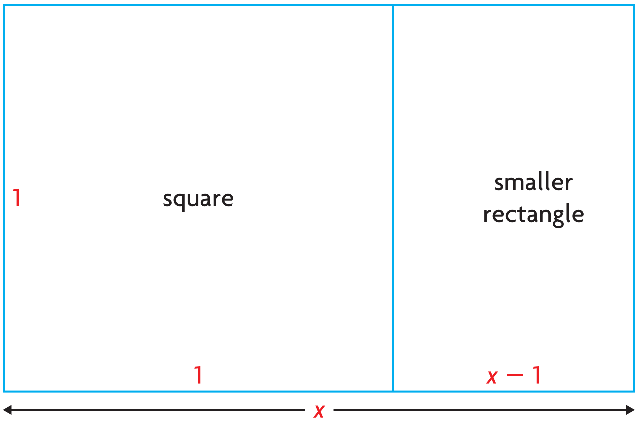
Curious Math

The Golden Rectangle

The golden rectangle is considered one of the most pleasing shapes to the human eye. It is often used in architectural design, and it can be seen in many famous works of art. For example, the golden rectangle can be seen in Leonardo Da Vinci's *Mona Lisa* and in the *Parthenon* in Athens, Greece.

One of the properties of the golden rectangle is its dimensions. When it is divided into a square and a smaller rectangle, the smaller rectangle is similar to the original rectangle.

Golden Rectangle



The ratio of the longer side to the shorter side in a golden rectangle is called the golden ratio.

If the length of the shorter side is 1 unit, and if x represents the length of the longer side, then $\frac{x}{1} = x$ is also the value of the golden ratio. A quadratic relation can be used to determine the value of the golden ratio.

1. Create a proportion statement to compare the golden ratio with the ratio of the lengths of the corresponding sides in the smaller rectangle.
2. Substitute the values in the diagram into your proportion statement. Then rearrange your proportion statement to obtain a quadratic relation.
3. Using graphing technology, graph the quadratic relation that corresponds to this equation.
4. What feature of the graph represents the value of the golden ratio?
5. Use graphing technology to determine the value of the golden ratio, correct to three decimal places.

5.6 Connecting Standard and Vertex Forms

GOAL

Sketch or graph a quadratic relation with an equation of the form $y = ax^2 + bx + c$ using symmetry.

YOU WILL NEED

- grid paper
- ruler

INVESTIGATE the Math

Many places hold a fireworks display on Canada Day. Clayton, a member of the local fire department, launches a series of rockets from a barge that is floating in the middle of the lake. Each rocket is choreographed to explode at the correct time. The equation $h = -5t^2 + 40t + 2$ can be used to model the height, h, of each rocket in metres above the water at t seconds after its launch. A certain rocket is scheduled to explode 3 min 21 s into the program.

Safety Connection

Fireworks can cause serious injury when handled incorrectly. It is safer to watch a community display than to create your own.

? Assuming that the rocket will explode at its highest point, when should Clayton launch it from the barge so it will explode at the correct time?

A. What information do you need to determine so that you can model the height of the rocket?

B. Copy and complete the table of values at the right for the rocket. Then plot the points, and sketch the graph of this relation.

Time (s)	Height (m)
0	2
1	37
2	
3	
4	
5	
6	
7	
8	
9	

C. What happens to the rocket between 8 s and 9 s after it is launched?

D. The axis of symmetry of a quadratic relation can be determined from the zeros. In this problem, however, there is only one zero because $t > 0$. Suggest another way to determine the axis of symmetry.

E. The rocket is 2 m above the water when it is launched. When will the rocket be at the same height again? Write the coordinates of these two points.

F. Consider the coordinates of the two points for part E. Why must the axis of symmetry be the same distance from both of these points? What is the equation of the axis of symmetry?

G. How does knowing the equation of the axis of symmetry help you determine the vertex of a parabola? What is the vertex of this parabola?

H. When should Clayton launch the rocket to ensure that it explodes 3 min 21 s into the program?

Reflecting

I. Can you determine the maximum height of the rocket directly from the standard form of the quadratic relation $h = -5t^2 + 40t + 2$? Explain.

J. How did you determine the vertex, even though one of the zeros of the quadratic relation was unknown?

K. Write the quadratic relation in vertex form. How can you compare this equation with the equation given in standard form to determine whether they are identical?

APPLY the Math

EXAMPLE 1 **Connecting** the vertex form to partial factors of the equation

Determine the maximum value of the quadratic relation $y = -3x^2 + 12x + 29$.

Michelle's Solution

$$y = -3x^2 + 12x + 29$$

I tried to factor the expression, but I couldn't determine two integers with a product of $(-3) \times 29$ and a sum of 12. This means that I can't use the zeros to help me.

$$y = x(-3x + 12) + 29$$

I had to determine the axis of symmetry, since the vertex (where the maximum value occurs) lies on it. To do this, I had to locate two points with the same *y*-coordinate. I removed a partial factor of *x* from the first two terms.

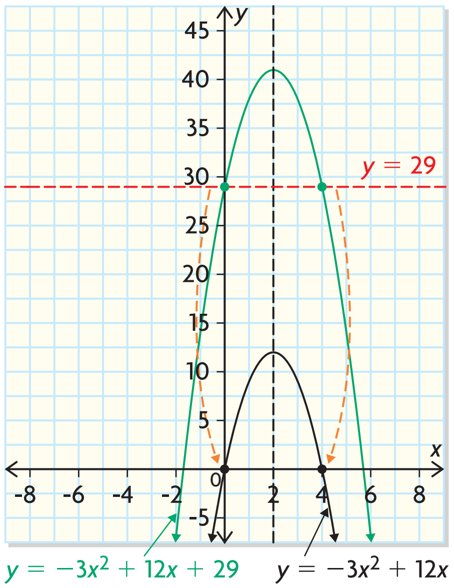
When $y = 29$,

$$29 = x(-3x + 12) + 29$$

$$x = 0 \text{ or } -3x + 12 = 0$$

$$x = 0 \qquad\qquad x = 4$$

I noticed that the *y*-value will be 29 if either factor in the equation equals 0. I decided to determine the two points on the parabola with a *y*-coordinate of 29 by setting each partial factor equal to 0 and solving for *x*.



I knew that this would work because it was like translating the graph down 29 units to make the points with a *y*-coordinate of 29 turn into points with a *y*-coordinate of 0.

This let me determine the zeros of the new translated graph.

I noticed that the axis of symmetry was the same for the two graphs.

(0, 29) and (4, 29) are on the graph of the original quadratic relation. This makes sense since the points (0, 0) and (4, 0) are the zeros of the translated graph.

These points are the same distance from the axis of symmetry. So, I know that the axis of symmetry is halfway between $x = 0$ and $x = 4$.

The equation of the axis of symmetry is

$$x = \frac{0 + 4}{2} \text{ so } x = 2.$$

I calculated the mean of the *x*-coordinates of these points to determine the axis of symmetry.

$$y = -3(2)^2 + 12(2) + 29$$
$$y = -12 + 24 + 29$$
$$y = 41$$

The *y*-coordinate of the vertex is the maximum value because the graph opens downward. To determine the maximum, I substituted $x = 2$ into $y = -3x^2 + 12x + 29$.

The maximum value is 41.

EXAMPLE 2 Selecting a partial factoring strategy to sketch the graph of a quadratic relation

Express the quadratic relation $y = 2x^2 + 8x + 5$ in vertex form.
Then sketch a graph of the relation by hand.

Marnie's Solution

$$y = 2x^2 + 8x + 5$$
$$y = x(2x + 8) + 5$$

This equation cannot be factored fully since you can't determine two integers with a product of 2×5 and a sum of 8. I removed a partial factor of *x* from the first two terms.

$$x = 0 \text{ or } 2x + 8 = 0$$
$$x = 0 \qquad x = -4$$

I found two points with a *y*-coordinate of 5 by setting each partial factor equal to 0. Both of these points are the same distance from the axis of symmetry.

The points (0, 5) and (−4, 5) are on the parabola.

The axis of symmetry is $x = -2$.

I found the axis of symmetry by calculating the mean of the *x*-coordinates of these points.

At the vertex,

$y = 2(-2)^2 + 8(-2) + 5$

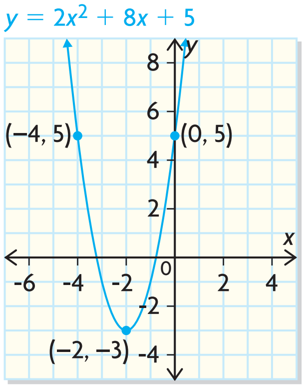
$y = -3$

Since the parabola is symmetric, the vertex is on the line $x = -2$. I substituted this value into the relation.

The vertex of the parabola is at $(-2, -3)$.

In vertex form, the equation of the parabola is $y = 2(x + 2)^2 - 3$.

I know that the value of *a* is the same in standard form and vertex form. In this case, $a = 2$.



The parameter *a* is positive, so the parabola opens upward.

I used the vertex and the two points I found, (0, 5) and (−4, 5), to sketch the parabola.

In Summary

Key Idea

- If a quadratic relation is in standard form and cannot be factored fully, you can use partial factoring to help you determine the axis of symmetry of the parabola. Then you can use the axis of symmetry to determine the coordinates of the vertex.

Need to Know

- If $y = ax^2 + bx + c$ cannot be factored, you can express the relation in the partially factored form $y = x(ax + b) + c$. Then you can use this form to determine the vertex form:
  - Set $x(ax + b) = 0$ and solve for *x* to determine two points on the parabola that are the same distance from the axis of symmetry. Both of these points have *y*-coordinate *c*.
  - Determine the axis of symmetry, $x = h$, by calculating the mean of the *x*-coordinates of these points.
  - Substitute $x = h$ into the relation to determine *k*, the *y*-coordinate of the vertex.
  - Substitute the values of *a*, *h*, and *k* into $y = a(x - h)^2 + k$.

CHECK Your Understanding

1. Determine the equation of the axis of symmetry of a parabola that passes through points (2, 8) and (−6, 8).

2. Determine two points that are the same distance from the axis of symmetry of the quadratic relation $y = 4x^2 - 12x + 5$.

3. Use partial factoring to determine the vertex form of the quadratic relation $y = 2x^2 - 10x + 11$.

PRACTISING

4. A parabola passes through points (3, 0), (7, 0), and (9, −24).

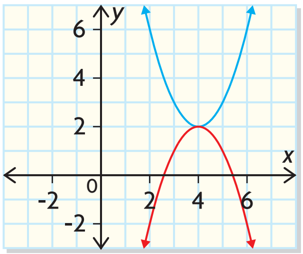
- **a)** Determine the equation of the axis of symmetry.
- **b)** Determine the coordinates of the vertex, and write the equation in vertex form.
- **c)** Write the equation in standard form.

5. For each quadratic relation,

- **i)** determine the coordinates of two points on the graph that are the same distance from the axis of symmetry
- **ii)** determine the equation of the axis of symmetry
- **iii)** determine the coordinates of the vertex
- **iv)** write the relation in vertex form

- **a)** $y = (x - 1)(x + 7)$
- **b)** $y = x(x - 6) - 8$
- **c)** $y = -2(x + 3)(x - 7)$
- **d)** $y = x(3x + 12) + 2$
- **e)** $y = x^2 + 5x$
- **f)** $y = x^2 - 11x + 21$

6. The equation of one of these parabolas at the right is $y = x^2 - 8x + 18$.
K Determine the equation of the other in vertex form.



7. For each quadratic relation,

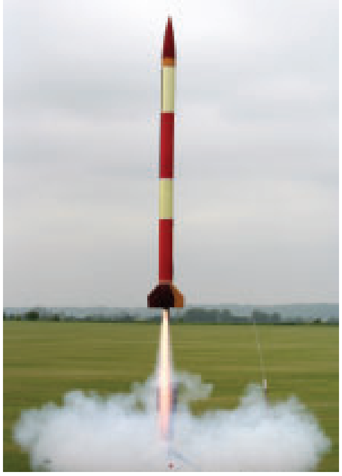
- **i)** use partial factoring to determine two points that are the same distance from the axis of symmetry
- **ii)** determine the coordinates of the vertex
- **iii)** express the relation in vertex form
- **iv)** sketch the graph

- **a)** $y = x^2 - 6x + 5$
- **b)** $y = x^2 - 4x - 11$
- **c)** $y = -2x^2 + 12x - 11$
- **d)** $y = -x^2 - 6x - 13$
- **e)** $y = -\frac{1}{2}x^2 + 2x - 3$
- **f)** $y = 2x^2 - 10x + 11$

8. Use two different strategies to determine the equation of the axis
C of symmetry of the parabola defined by $y = -2x^2 + 16x - 24$. Which strategy do you prefer? Explain why.

9. T Determine the values of a and b in the relation $y = ax^2 + bx + 7$ if the vertex is located at $(4, -5)$.

10. Determine the values of a and b in the relation $y = ax^2 + bx + 8$ if the vertex is located at $(1, 7)$.



11. A model rocket is launched straight up, with an initial velocity of 150 m/s. The height of the rocket can be modelled by $h = -5t^2 + 150t$, where h is the height in metres and t is the elapsed time in seconds. What is the maximum height reached by the rocket?

12. A A baseball is hit from a height of 1 m. The height, h, of the ball in metres after t seconds can be modelled by $h = -5t^2 + 9t + 1$. Determine the maximum height reached by the ball.

13. A movie theatre can accommodate a maximum of 450 moviegoers per day. The theatre operators have determined that the profit per day, P, is related to the ticket price, t, by $P = -30t^2 + 450t - 790$. What ticket price will maximize the daily profit?

14. The world production of gold from 1970 to 1990 can be modelled by $G = 1492 - 76t + 5.2t^2$, where G is the number of tonnes of gold and t is the number of years since 1970 ($t = 0$ for 1970, $t = 1$ for 1971, and so on).

a) During this period, when was the minimum amount of gold mined?

b) What was the least amount of gold mined in one year?

c) How much gold was mined in 1985?

15. Create a concept web that summarizes the different algebraic strategies you can use to determine the axis of symmetry and the vertex of a quadratic relation given in the form $y = ax^2 + bx + c$.

Extending

16. A farmer has \$3000 to spend on fencing for two adjoining rectangular pastures, both with the same dimensions. A local contracting company can build the fence for \$5.00/m. What is the largest total area that the farmer can have fenced for this price?

17. A city transit system carries an average of 9450 people per day on its buses, at a fare of \$1.75 each. The city wants to maximize the transit system's revenue by increasing the fare. A survey shows that the number of riders will decrease by 210 for every \$0.05 increase in the fare. What fare will result in the greatest revenue? How many daily riders will they lose at this new fare?

5 Chapter Review

FREQUENTLY ASKED Questions

Q: What information do I need about the graph of a quadratic relation to write its equation in vertex form?

A: You can write the equation in vertex form if you know the coordinates of the vertex and one additional point on the graph.

Study Aid

- See Lesson 5.4, Examples 1 to 3.
- Try Chapter Review Questions 8 to 10.

Q: What kinds of problems can you solve using the vertex form of a quadratic relation?

A: The vertex form of a quadratic relation is usually the easiest form to use when you need to determine the equation of a parabola of good fit on a scatter plot. The vertex form is also useful when you need to determine the maximum or minimum value of a quadratic relation.

Study Aid

- See Lesson 5.5, Examples 1, 2, and 4.
- Try Chapter Review Questions 11 to 13.

Q: How can you relate the standard form of a quadratic relation to its vertex form?

A: The standard form of a quadratic relation, $y = ax^2 + bx + c$, can be rewritten in vertex form if you know the value of a and the coordinates of the vertex:

- If the quadratic relation can be written in factored form, you can determine the zeros by setting each factor equal to zero. Calculating the mean of the x-coordinates of the zeros gives you the axis of symmetry and the x-coordinate of the vertex. Substitute the x-coordinate of the vertex into the quadratic relation to determine the y-coordinate of the vertex.
- If $y = ax^2 + bx + c$ cannot be factored, you can use partial factoring to express the equation in the form $y = x(ax + b) + c$. Solving $x(ax + b) = 0$ gives two points with the same y-coordinate, c. Calculating the mean of the x-coordinates of these points gives you the axis of symmetry and the x-coordinate of the vertex. Substitute the x-coordinate of the vertex into the quadratic relation to determine the y-coordinate of the vertex.
- If you graph $y = ax^2 + bx + c$ using graphing technology, then you can approximate the vertex from the graph or determine it exactly, depending on the features of the technology.

In all cases, after you know the vertex, you can use the value of a from the standard form of the relation to write the relation in vertex form, $y = a(x - h)^2 + k$.

Study Aid

- See Lesson 5.5, Example 3, and Lesson 5.6, Examples 1 and 2.
- Try Chapter Review Questions 14 to 17.

PRACTICE Questions

Lesson 5.1

1. Write the equations of two different quadratic relations that match each description.

a) The graph has a narrower opening than the graph of $y = 2x^2$.

b) The graph has a wider opening than the graph of $y = -0.5x^2$.

c) The graph opens downward and has a narrower opening than the graph of $y = 5x^2$.

2. The point (p, q) lies on the parabola $y = ax^2$. If you did not know the value of a, how could you use the values of p and q to determine whether the parabola is wider or narrower than $y = x^2$?

Lesson 5.2

3. Match each translation with the correct quadratic relation.

a) 3 units left, 4 units down

b) 2 units right, 4 units down

c) 5 units left

d) 3 units right, 2 units up

i) $y = (x - 3)^2 + 2$ **iii)** $y = (x - 2)^2 - 4$

ii) $y = (x + 3)^2 - 4$ **iv)** $y = (x + 5)^2$

Lesson 5.3

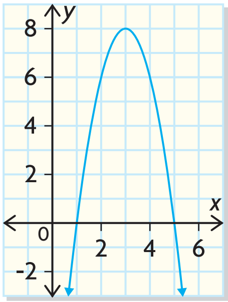
4. Which equation represents the graph shown? Explain your reasoning.

a) $y = -3(x + 3)^2 + 8$

b) $y = -3(x - 3)^2 + 8$

c) $y = 3(x - 3)^2 - 8$

d) $y = -2(x - 3)^2 + 8$



5. The parabola $y = x^2$ is transformed in two different ways to produce the parabolas $y = 2(x - 4)^2 + 5$ and $y = 2(x - 5)^2 + 4$. How are these transformations the same, and how are they different?

6. Blake rotated the parabola $y = x^2$ by 180° around a point. The new vertex is $(6, -8)$. What is the equation of the new parabola?

7. Reggie used transformations to graph $y = -2(x - 4)^2 + 3$. He started by reflecting the graph of $y = x^2$ in the x-axis. Then he translated the graph so that its vertex moved to $(4, 3)$. Finally, he stretched the graph vertically by a factor of 2.

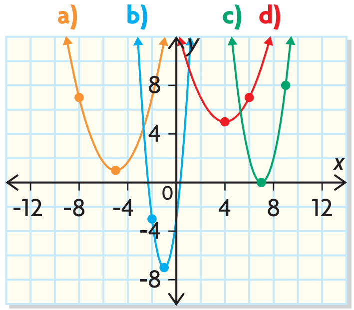
a) Why was Reggie's final graph not correct?

b) What sequence of transformations should he have used?

c) Use transformations to sketch $y = -2(x - 4)^2 + 3$ on grid paper.

Lesson 5.4

8. Use the point marked on each parabola, as well as the vertex of the parabola, to determine the equation of the parabola in vertex form.



9. Use the given information to determine the equation of each quadratic relation in vertex form.

a) vertex at $(-3, 2)$, passes through $(-1, 4)$

b) vertex at $(1, 5)$, passes through $(3, -3)$

10. This table shows residential energy use by Canadians from 2002 to 2006, where 1 petajoule equals 1 000 000 000 000 000 joules.

Year	Residential Energy Use (petajoules)
2002	1286.70
2003	1338.20
2004	1313.00
2005	1296.60
2006	1250.30

a) Use technology to create a scatter plot and a quadratic regression model.
b) Determine the vertex, and write the equation of the model in vertex form.
c) According to your model, when was energy use at a maximum during this period?

Lesson 5.5

11. Karla hits a golf ball from an elevated tee to the green below. This table shows the height of the ball above the ground as it moves through the air.

Time (s)	Height (m)
0.0	30.00
0.5	41.25
1.0	50.00
1.5	56.25
2.0	60.00
2.5	61.25
3.0	60.00
3.5	56.25
4.0	50.00

a) Create a scatter plot, and draw a curve of good fit.
b) Estimate the coordinates of the vertex.
c) Determine a quadratic relation in vertex form to model the data.
d) Use the quadratic regression feature of graphing technology to create a model for the data. Compare this model with the model you created by hand for part c). How accurate is the model you created by hand?

12. A farming community collected data on the effect of different amounts of fertilizer, x, in 100 kg/ha, on the yield of carrots, y, in tonnes. The resulting quadratic regression model is $y = -0.5x^2 + 1.4x + 0.1$. Determine the amount of fertilizer needed to produce the maximum yield.

13. A local club alternates between booking live bands and booking DJs. By tracking receipts over a period of time, the owner of the club determined that her profit from a live band depended on the ticket price. Her profit, P, can be modelled using $P = -15x^2 + 600x + 50$, where x represents the ticket price in dollars.
a) Sketch the graph of the relation to help the owner understand this profit model.
b) Determine the maximum profit and the ticket price she should charge to achieve the maximum profit.

14. For each quadratic relation,
i) write the equation in factored form
ii) determine the coordinates of the vertex
iii) write the equation in vertex form
iv) sketch the graph

a) $y = x^2 - 8x + 15$
b) $y = 2x^2 - 8x - 64$
c) $y = -4x^2 - 12x + 7$

Lesson 5.6

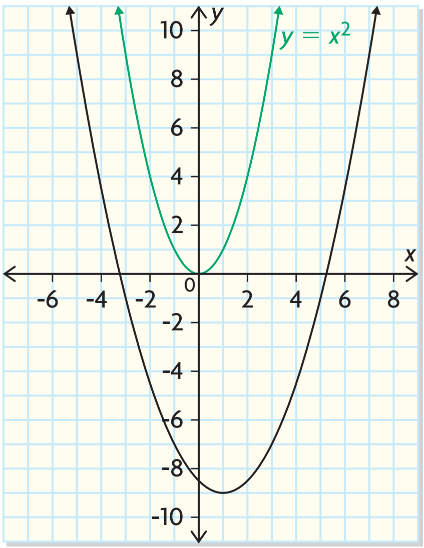
15. Express each quadratic relation in vertex form using partial factoring to determine two points that are the same distance from the axis of symmetry.
a) $y = x^2 + 2x + 5$
b) $y = -x^2 + 6x - 3$
c) $y = -3x^2 + 42x - 147$
d) $y = 2x^2 - 20x + 41$

16. Write each quadratic relation in vertex form using an appropriate strategy.
a) $y = x^2 - 6x - 8$
b) $y = -2(x + 3)(x - 7)$
c) $y = x(3x + 12) + 2$
d) $y = -2x^2 + 12x - 11$

17. The height, h, of a football in metres t seconds since it was kicked can be modelled by $h = -4.9t^2 + 22.54t + 1.1$.
a) What was the height of the football when the punter kicked it?
b) Determine the maximum height of the football, correct to one decimal place, and the time when it reached this maximum height.

5

Chapter Self-Test



1. **a)** The **black** graph at the left resulted from transforming the **green** graph of $y = x^2$. Determine the equation of the black graph. Explain your reasoning.
   b) State the transformations that were applied to the graph of $y = x^2$ to result in the black graph.

2. Determine the equation of each quadratic relation in vertex form.
   a) vertex at (7, 5), opens downward, vertical stretch of 4
   b) zeros at 1 and 5, minimum value of -12, passes through (6, 15)

3. Sketch each quadratic relation by applying the correct sequence of transformations to the graph of $y = x^2$.
   a) $y = -2(x - 3)^2 + 8$ **b)** $y = 0.5(x + 2)^2 - 5$

4. The parabola $y = x^2$ is compressed vertically and translated down and right. The point $(4, -10)$ is on the new graph. What is a possible equation for the new graph?

5. Accountants for the HiTech Shoe Company have determined that the quadratic relation $P = -2x^2 + 24x - 54$ models the company's profit for the next quarter. In this relation, P represents the profit (in $100 000s) and x represents the number of pairs of shoes sold (in 100 000s).
   a) Express the equation in factored form.
   b) What are the zeros of the relation? What do they represent in this context?
   c) Determine the number of pairs of shoes that the company must sell to maximize its profit. How much would the maximum profit be?

6. A toy rocket that is sitting on a tower is launched vertically upward. The table shows the height, h, of the rocket in centimetres at t seconds after its launch.

***t* (s)**	0	1	2	3	4	5	6	7
***h* (cm)**	88	107	116	115	104	83	52	11

   a) Using a graphing calculator, create a scatter plot to display the data.
   b) Estimate the vertex of your model. Then write the equation of the model in vertex form and standard form.
   c) Use the regression feature on the graphing calculator to create a quadratic model for the data. Compare this model with the model you created for part b).
   d) What is the maximum height of the rocket? When does the rocket reach this maximum height?
   e) When will the rocket hit the ground?

Process ***Checklist***

- ✔ Question 2: Did you **connect** the information about each parabola to the appropriate form of the relation?
- ✔ Question 4: Did you apply **reasoning** skills as you developed a possible equation for the graph?
- ✔ Questions 5 and 6: Did you **reflect** on your thinking to assess the appropriateness of your strategies as you solved the problems?
- ✔ Question 6: Did you relate the numeric, algebraic, graphical, and verbal **representations** of the situation?

Chapter Task

Human Immunodeficiency Virus (HIV)

Every year, many people become infected with HIV. Over 90% of HIV infections in children in the United States are due to mother-to-child transmission at birth. The data in the table show the number of mother-to-child HIV infections diagnosed in the United States in various years from 1985 to 2005.

Year	1985	1987	1990	1993	1996	1998	2001	2003	2005
Number of Cases	210	500	780	770	460	300	317	188	142

? What can we learn about the fight against HIV infections from the data?

A. Create a scatter plot to display the data. Why does a quadratic model make sense?

B. Determine an equation for this relation. Which form (standard, vertex, or factored) do you think is the best for these data? Explain.

C. Identify the transformations that you would apply to the graph of $y = x^2$ to obtain the model you created for part B.

D. The decline in the number of mother-to-child HIV infections is due to the introduction of preventive drug therapies. Based on your model, when do you think an effective drug therapy was first introduced? Explain your reasoning.

E. Based on your model, will the number of HIV cases ever be reduced to zero? If so, when might this occur? Do you think your prediction is accurate? Explain your reasoning.

F. Suggest some reasons why a mathematical description of the data could be useful to researchers or government agencies.

Health *Connection*

Research on HIV focuses on prevention, and on treatment and care of people infected. A mask and gloves are needed for protection.

Task *Checklist*

- ✓ Did you draw your graph accurately and label it correctly?
- ✓ Did you choose an appropriate graphing window to display your scatter plot?
- ✓ Did you show all the appropriate calculations?
- ✓ Did you explain your reasoning clearly?