

## Organized Counting With Venn Diagrams

In Chapter 4, you used tree diagrams as a tool for counting items when the order of the items was important. This section introduces a type of diagram that helps you organize data about groups of items when the order of the items is not important.

### INVESTIGATE & INQUIRE: Visualizing Relationships Between Groups

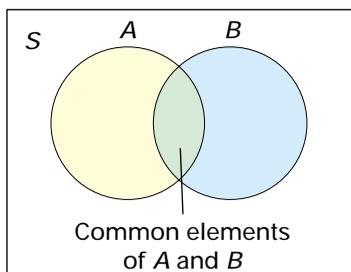
A group of students meet regularly to plan the dances at Vennville High School. Amar, Belinda, Charles, and Danica are on the dance committee, and Belinda, Charles, Edith, Franco, and Geoff are on the students' council. Hans and Irena are not members of either group, but they attend meetings as reporters for the school newspaper.

1. Draw two circles to represent the dance committee and the students' council. Where on the diagram would you put initials representing the students who are
  - a) on the dance committee?
  - b) on the students' council?
  - c) on the dance committee and the students' council?
  - d) not on either the dance committee or the students' council?
2. Redraw your diagram marking on it the number of initials in each region. What relationships can you see among these numbers?



Your sketch representing the dance committee and the students' council is a simple example of a **Venn diagram**. The English logician John Venn (1834–1923) introduced such diagrams as a tool for analysing situations where there is some overlap among groups of items, or **sets**. Circles represent different sets and a rectangular box around the circles represents the **universal set**,  $S$ , from which all the items are drawn. This box is usually labelled with an  $S$  in the top left corner.

The items in a set are often called the **elements** or **members** of the set. The size of a circle in a Venn diagram does not have to be proportional to the number of elements in the set the circle represents. When some items in a set are also elements of another set, these items are **common elements** and the sets are shown as overlapping circles. If *all* elements of a set  $C$  are also elements of set  $A$ , then  $C$  is a **subset** of  $A$ . A Venn diagram would show this set  $C$  as a region contained within the circle for set  $A$ .



*The common elements are a subset of both A and B.*

**WEB CONNECTION**  
  
[www.mcgrawhill.ca/links/MDM12](http://www.mcgrawhill.ca/links/MDM12)

To learn more about Venn diagrams, visit the above web site and follow the links. Describe an example of how Venn diagrams can be used to organize information.

You can use Venn diagrams to organize information for situations in which the number of items in a group are important but the order of the items is not.

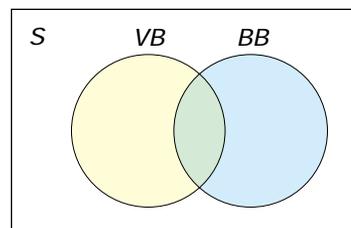
### Example 1 Common Elements

There are 10 students on the volleyball team and 15 on the basketball team. When planning a field trip with both teams, the coach has to arrange transportation for a total of only 19 students.

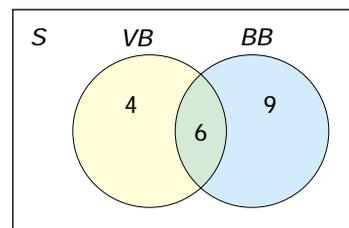
- Use a Venn diagram to illustrate this situation.
- Explain why you cannot use the additive counting principle to find the total number of students on the teams.
- Determine how many students are on both teams.
- Determine the number of students in the remaining regions of your diagram and explain what these regions represent.

#### Solution

- Some students must be on both the volleyball and the basketball team. Draw a box with an  $S$  in the top left-hand corner. Draw and label two overlapping circles to represent the volleyball and basketball teams.



- b) The additive counting principle (or rule of sum) applies only to mutually exclusive events or items. However, it is possible for students to be on both teams. If you simply add the 10 students on the volleyball team to 15 students on the basketball team, you get a total of 25 students because the students who play on both teams have been counted twice.
- c) The difference between the total in part b) and the total number of students actually on the two teams is equal to the number of students who are members of both teams. Thus,  $25 - 19 = 6$  students play on both teams. In the Venn diagram, these 6 students are represented by the area where the two circles overlap.
- d) There are  $10 - 6 = 4$  students in the section of the  $VB$  circle that does not overlap with the  $BB$  circle. These are the students who play only on the volleyball team. Similarly, the non-overlapping portion of the  $BB$  circle represents the  $15 - 6 = 9$  students who play only on the basketball team.



Example 1 illustrates the **principle of inclusion and exclusion**. If you are counting the total number of elements in two groups or sets that have common elements, you must subtract the common elements so that they are not included twice.

#### Principle of Inclusion and Exclusion for Two Sets

For sets  $A$  and  $B$ , the total number of elements in either  $A$  or  $B$  is the number in  $A$  plus the number in  $B$  minus the number in both  $A$  and  $B$ .

$$n(A \text{ or } B) = n(A) + n(B) - n(A \text{ and } B),$$

where  $n(X)$  represents the numbers of elements in a set  $X$ .

The set of all elements in either set  $A$  or set  $B$  is the **union** of  $A$  and  $B$ , which is often written as  $A \cup B$ . Similarly, the set of all elements in both  $A$  and  $B$  is the **intersection** of  $A$  and  $B$ , written as  $A \cap B$ . Thus the principle of inclusion and exclusion for two sets can also be stated as

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

Note that the additive counting principle (or rule of sum) could be considered a special case of the principle of inclusion and exclusion that applies only when sets  $A$  and  $B$  have no elements in common, so that  $n(A \text{ and } B) = 0$ . The principle of inclusion and exclusion can also be applied to three or more sets.



### Example 2 Applying the Principle of Inclusion and Exclusion

A drama club is putting on two one-act plays. There are 11 actors in the Feydeau farce and 7 in the Molière piece.

- If 3 actors are in both plays, how many actors are there in all?
- Use a Venn diagram to calculate how many students are in only one of the two plays.

#### Solution

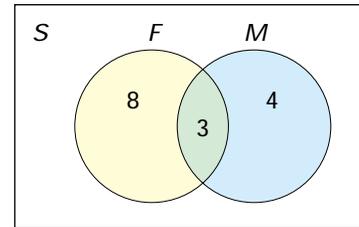
- Calculate the number of students in both plays using the principle of inclusion and exclusion.

$$\begin{aligned}n(\text{total}) &= n(\text{Feydeau}) + n(\text{Molière}) - n(\text{Feydeau and Molière}) \\ &= 11 + 7 - 3 \\ &= 15\end{aligned}$$

There are 15 students involved in the two one-act plays.

- There are 3 students in the overlap between the two circles. So, there must be  $11 - 3 = 8$  students in the region for Feydeau only and  $7 - 3 = 4$  students in the region for Molière only.

Thus, a total of  $8 + 4 = 12$  students are in only one of the two plays.



As in the first example, using a Venn diagram can clarify the relationships between several sets and subsets.



### Example 3 Working With Three Sets

Of the 140 grade 12 students at Vennville High School, 52 have signed up for biology, 71 for chemistry, and 40 for physics. The science students include 15 who are taking both biology and chemistry, 8 who are taking chemistry and physics, 11 who are taking biology and physics, and 2 who are taking all three science courses.

- How many students are not taking any of these three science courses?
- Illustrate the enrolments with a Venn diagram.

#### Solution

- Extend the principle of inclusion and exclusion to three sets. Total the numbers of students in each course, subtract the numbers of students taking two courses, then add the number taking all three. This procedure subtracts out the students who have been counted twice because they are in two

courses, and then adds back those who were subtracted twice because they were in all three courses.

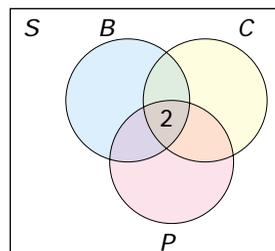
For simplicity, let  $B$  stand for biology,  $C$  stand for chemistry, and  $P$  stand for physics. Then, the total number of students taking at least one of these three courses is

$$\begin{aligned} n(\text{total}) &= n(B) + n(C) + n(P) - n(B \text{ and } C) - n(C \text{ and } P) - n(B \text{ and } P) + n(B \text{ and } C \text{ and } P) \\ &= 52 + 71 + 40 - 15 - 8 - 11 + 2 \\ &= 131 \end{aligned}$$

There are 131 students taking one or more of the three science courses. To find the number of grade 12 students who are not taking any of these science courses, subtract 131 from the total number of grade 12 students.

Thus,  $140 - 131 = 9$  students are not taking any of these three science courses in grade 12.

- b)** For this example, it is easiest to start with the overlap among the three courses and then work outward. Since there are 2 students taking all three courses, mark 2 in the centre of the diagram where the three circles overlap.

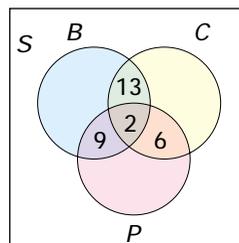


Next, consider the adjacent regions representing the students who are taking exactly two of the three courses.

*Biology and chemistry:* Of the 15 students taking these two courses, 2 are also taking physics, so 13 students are taking only biology and chemistry.

*Chemistry and physics:* 8 students less the 2 in the centre region leaves 6.

*Biology and physics:*  $11 - 2 = 9$ .



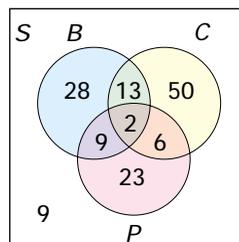
Now, consider the regions representing students taking only one of the science courses.

*Biology:* Of the 52 students taking this course,  $13 + 2 + 9 = 24$  are in the regions overlapping with the other two courses, leaving 28 students who are taking biology only.

*Chemistry:* 71 students less the  $13 + 2 + 6$  leaves 50.

*Physics:*  $40 - (9 + 2 + 6) = 23$ .

Adding all the numbers within the circles gives a total of 131. Thus, there must be  $140 - 131 = 9$  grade 12 students who are not taking any of the three science courses, which agrees with the answer found in part a).



## Key Concepts

- Venn diagrams can help you visualize the relationships between sets of items, especially when the sets have some items in common.
- The principle of inclusion and exclusion gives a formula for finding the number of items in the union of two or more sets. For two sets, the formula is  $n(A \text{ or } B) = n(A) + n(B) - n(A \text{ and } B)$ .

## Communicate Your Understanding

1. Describe the principal use of Venn diagrams.
2. Is the universal set the same for all Venn diagrams? Explain why or why not.
3. Explain why the additive counting principle can be used in place of the principle of inclusion and exclusion for mutually exclusive sets.

## Practise

A

1. Let set  $A$  consist of an apple, an orange, and a pear and set  $B$  consist of the apple and a banana.

a) List the elements of

- i)  $A$  and  $B$
- ii)  $A$  or  $B$
- iii)  $S$
- iv)  $S \cap B$
- v)  $A \cup B \cup S$

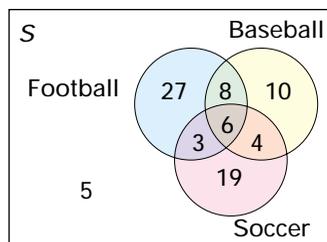
b) List the value of

- i)  $n(A) + n(B)$
- ii)  $n(A \text{ or } B)$
- iii)  $n(S)$
- iv)  $n(A \cup B)$
- v)  $n(S \cap A)$

c) List all subsets containing exactly two elements for

- i)  $A$
- ii)  $B$
- iii)  $A \cup B$

2. A recent survey of a group of students found that many participate in baseball, football, and soccer. The Venn diagram below shows the results of the survey.



- a) How many students participated in the survey?
- b) How many of these students play both soccer and baseball?
- c) How many play only one sport?
- d) How many play football and soccer?
- e) How many play all three sports?
- f) How many do not play soccer?

## Apply, Solve, Communicate

### B

3. Of the 220 graduating students in a school, 110 attended the semi-formal dance and 150 attended the formal dance. If 58 students attended both events, how many graduating students did not attend either of the two dances? Illustrate your answer with a Venn diagram.
4. **Application** A survey of 1000 television viewers conducted by a local television station produced the following data:
  - 40% watch the news at 12 00
  - 60% watch the news at 18 00
  - 50% watch the news at 23 00
  - 25% watch the news at 12 00 and at 18 00
  - 20% watch the news at 12 00 and at 23 00
  - 20% watch the news at 18 00 and at 23 00
  - 10% watch all three news broadcasts
  - a) What percent of those surveyed watch at least one of these programs?
  - b) What percent watch none of these news broadcasts?
  - c) What percent view the news at 12 00 and at 18 00, but not at 23 00?
  - d) What percent view only one of these shows?
  - e) What percent view exactly two of these shows?
5. Suppose the Canadian Embassy in the Netherlands has 32 employees, all of whom speak both French and English. In addition, 22 of the employees speak German and 15 speak Dutch. If there are 10 who speak both German and Dutch, how many of the employees speak neither German nor Dutch? Illustrate your answer with a Venn diagram.
6. **Application** There are 900 employees at CantoCrafts Inc. Of these, 615 are female, 345 are under 35 years old, 482 are single, 295 are single females, 187 are singles under 35 years old, 190 are females under 35 years old, and 120 are single females under 35 years old. Use a Venn diagram to determine how many employees are married males who are at least 35 years old.
7. **Communication** A survey of 100 people who volunteered information about their reading habits showed that
  - 75 read newspapers daily
  - 35 read books at least once a week
  - 45 read magazines regularly
  - 25 read both newspapers and books
  - 15 read both books and magazines
  - 10 read newspapers, books, and magazines
  - a) Construct a Venn diagram to determine the maximum number of people in the survey who read both newspapers and magazines.
  - b) Explain why you cannot determine exactly how many of the people surveyed read both newspapers and magazines.

8. Jeffrey works as a DJ at a local radio station. On occasion, he chooses some of the songs he will play based on the phone-in requests received by the switchboard the previous day. Jeffrey's list of 200 possible selections includes

- all the songs in the top 100
- 134 hard-rock songs
- 50 phone-in requests
- 45 hard-rock songs in the top 100
- 20 phone-in requests in the top 100
- 24 phone-in requests for hard-rock songs

Use a Venn diagram to determine

- a) how many phone-in requests were for hard-rock songs in the top 100
- b) how many of the songs in the top 100 were neither phone-in requests nor hard-rock selections

## C

9. **Inquiry/Problem Solving** The Vennville junior hockey team has 12 members who can play forward, 8 who can play defence, and 2 who can be goalies. What is the smallest possible size of the team if
- a) no one plays more than one position?
  - b) no one plays both defence and forward?
  - c) three of the players are able to play defence or forward?
  - d) both the goalies can play forward but not defence?
10. **Inquiry/Problem Solving** Use the principle of inclusion and exclusion to develop a formula for the number of elements in
- a) three sets    b) four sets    c)  $n$  sets

## Career Connection

## Forensic Scientist

The field of forensic science could be attractive to those with a mathematics and science background. The job of a forensic scientist is to identify, analyse, and match items collected from crime scenes.

Forensic scientists most often work in a forensic laboratory. Such laboratories examine and analyse physical evidence, including controlled substances, biological materials, firearms and ammunition components, and DNA samples.

Forensic scientists may have specialities such as fingerprints, ballistics, clothing and fibres, footprints, tire tracks, DNA profiling, or crime scene analysis.

Modern forensic science combines mathematics and computers. A forensic scientist should have a background in combinatorics, biology, and the physical sciences. Forensic scientists work for a wide variety of organizations including police forces, government offices, and the military.

## WEB CONNECTION

[www.mcgrawhill.ca/links/MDM12](http://www.mcgrawhill.ca/links/MDM12)

For more information about forensic science and other careers related to mathematics, visit the above web site and follow the links. Write a brief description of how combinatorics could be used by forensic scientists.