5.4 The Binomial Theorem

Blaise Pascal

Recall that a binomial is a polynomial with just two terms, so it has the form $a + b$. Expanding $(a + b)^n$ becomes very laborious as n increases. This section introduces a method for expanding powers of binomials. This method is useful both as an algebraic tool and for probability calculations, as you will see in later chapters.

INVESTIGATE & INQUIRE: Patterns in the Binomial Expansion

1. Expand each of the following and simplify fully.
   a) $(a + b)^1$ b) $(a + b)^2$ c) $(a + b)^3$
   d) $(a + b)^4$ e) $(a + b)^5$
2. Study the terms in each of these expansions. Describe how the degree of each term relates to the power of the binomial.
3. Compare the terms in Pascal's triangle to the expansions in question 1. Describe any pattern you find.
4. Predict the terms in the expansion of $(a + b)^6$.

In section 4.4, you found a number of patterns in Pascal's triangle. Now that you are familiar with combinations, there is another important pattern that you can recognize. Each term in Pascal's triangle corresponds to a value of ${}_nC_r$.

$$\begin{array}{c} 1 \\ 1 \quad 1 \\ 1 \quad 2 \quad 1 \\ 1 \quad 3 \quad 3 \quad 1 \\ 1 \quad 4 \quad 6 \quad 4 \quad 1 \\ 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \end{array}$$

$$\begin{array}{c} {}_0C_0 \\ {}_1C_0 \quad {}_1C_1 \\ {}_2C_0 \quad {}_2C_1 \quad {}_2C_2 \\ {}_3C_0 \quad {}_3C_1 \quad {}_3C_2 \quad {}_3C_3 \\ {}_4C_0 \quad {}_4C_1 \quad {}_4C_2 \quad {}_4C_3 \quad {}_4C_4 \\ {}_5C_0 \quad {}_5C_1 \quad {}_5C_2 \quad {}_5C_3 \quad {}_5C_4 \quad {}_5C_5 \end{array}$$

Comparing the two triangles shown on page 289, you can see that $t_{n,r} = {}_nC_r$. Recall that Pascal's method for creating his triangle uses the relationship

$t_{n,r} = t_{n-1,\,r-1} + t_{n-1,\,r}$.

So, this relationship must apply to combinations as well.

Pascal's Formula

$${}_nC_r = {}_{n-1}C_{r-1} + {}_{n-1}C_r$$

Proof:

$$\begin{aligned}
{}_{n-1}C_{r-1} + {}_{n-1}C_r &= \frac{(n-1)!}{(r-1)!(n-r)!} + \frac{(n-1)!}{r!(n-r-1)!} \\
&= \frac{r(n-1)!}{r(r-1)!(n-r)!} + \frac{(n-1)!(n-r)}{r!(n-r)(n-r-1)!} \\
&= \frac{r(n-1)!}{r!(n-r)!} + \frac{(n-1)!(n-r)}{r!(n-r)!} \\
&= \frac{(n-1)!}{r!(n-r)!}[r + (n-r)] \\
&= \frac{(n-1)! \times n}{r!(n-r)!} \\
&= \frac{n!}{r!(n-r)!} \\
&= {}_nC_r
\end{aligned}$$

This proof shows that the values of ${}_nC_r$ do indeed follow the pattern that creates Pascal's triangle. It follows that ${}_nC_r = t_{n,r}$ for all the terms in Pascal's triangle.

Example 1 Applying Pascal's Formula to Combinations

Rewrite each of the following using Pascal's formula.

a) ${}_{12}C_8$

b) ${}_{19}C_5 + {}_{19}C_6$

Solution

a) ${}_{12}C_8 = {}_{11}C_7 + {}_{11}C_8$

b) ${}_{19}C_5 + {}_{19}C_6 = {}_{20}C_6$

As you might expect from the investigation at the beginning of this section, the coefficients of each term in the expansion of $(a + b)^n$ correspond to the terms in row n of Pascal's triangle. Thus, you can write these coefficients in combinatorial form.

The Binomial Theorem

$$(a+b)^n = {}_nC_0a^n + {}_nC_1a^{n-1}b + {}_nC_2a^{n-2}b^2 + \ldots + {}_nC_ra^{n-r}b^r + \ldots + {}_nC_nb^n$$
$$\text{or } (a+b)^n = \sum_{r=0}^{n} {}_nC_ra^{n-r}b^r$$

Example 2 Applying the Binomial Theorem

Use combinations to expand $(a + b)^6$.

Solution

$$\begin{aligned}(a+b)^6 &= \sum_{r=0}^{6} {}_6C_ra^{6-r}b^r\\ &= {}_6C_0a^6 + {}_6C_1a^5b + {}_6C_2a^4b^2 + {}_6C_3a^3b^3 + {}_6C_4a^2b^4 + {}_6C_5ab^5 + {}_6C_6b^6\\ &= a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6\end{aligned}$$

Example 3 Binomial Expansions Using Pascal's Triangle

Use Pascal's triangle to expand

a) $(2x - 1)^4$

b) $(3x - 2y)^5$

Solution

a) Substitute $2x$ for a and -1 for b. Since the exponent is 4, use the terms in row 4 of Pascal's triangle as the coefficients: 1, 4, 6, 4, and 1. Thus,

$$\begin{aligned}(2x-1)^4 &= 1(2x)^4 + 4(2x)^3(-1) + 6(2x)^2(-1)^2 + 4(2x)(-1)^3 + 1(-1)^4\\ &= 16x^4 + 4(8x^3)(-1) + 6(4x^2)(1) + 4(2x)(-1) + 1\\ &= 16x^4 - 32x^3 + 24x^2 - 8x + 1\end{aligned}$$

b) Substitute $3x$ for a and $-2y$ for b, and use the terms from row 5 as coefficients.

$$\begin{aligned}(3x-2y)^5 &= 1(3x)^5 + 5(3x)^4(-2y) + 10(3x)^3(-2y)^2 + 10(3x)^2(-2y)^3 + 5(3x)(-2y)^4 + 1(-2y)^5\\ &= 243x^5 - 810x^4y + 1080x^3y^2 - 720x^2y^3 + 240xy^4 - 32y^5\end{aligned}$$

Example 4 Expanding Binomials Containing Negative Exponents

Use the binomial theorem to expand and simplify $\left(x + \frac{2}{x^2}\right)^4$.

Solution

Substitute x for a and $\frac{2}{x^2}$ for b.

$$\left(x + \frac{2}{x^2}\right)^4 = \sum_{r=0}^{4} {}_4C_r x^{4-r}\left(\frac{2}{x^2}\right)^r$$

$$= {}_4C_0x^4 + {}_4C_1x^3\left(\frac{2}{x^2}\right) + {}_4C_2x^2\left(\frac{2}{x^2}\right)^2 + {}_4C_3x\left(\frac{2}{x^2}\right)^3 + {}_4C_4\left(\frac{2}{x^2}\right)^4$$

$$= 1x^4 + 4x^3\left(\frac{2}{x^2}\right) + 6x^2\left(\frac{4}{x^4}\right) + 4x\left(\frac{8}{x^6}\right) + 1\left(\frac{16}{x^8}\right)$$

$$= x^4 + 8x + 24x^{-2} + 32x^{-5} + 16x^{-8}$$

Example 5 Patterns With Combinations

Using the patterns in Pascal's triangle from the investigation and Example 4 in section 4.4, write each of the following in combinatorial form.

a) the sum of the terms in row 5 and row 6

b) the sum of the terms in row n

c) the first 5 triangular numbers

d) the nth triangular number

Solution

a) *Row 5:*

$$1 + 5 + 10 + 10 + 5 + 1$$
$$= {}_5C_0 + {}_5C_1 + {}_5C_2 + {}_5C_3 + {}_5C_4 + {}_5C_5$$
$$= 32$$
$$= 2^5$$

Row 6:

$$1 + 6 + 15 + 20 + 15 + 6 + 1$$
$$= {}_6C_0 + {}_6C_1 + {}_6C_2 + {}_6C_3 + {}_6C_4 + {}_6C_5 + {}_6C_6$$
$$= 64$$
$$= 2^6$$

b) From part a) it appears that ${}_nC_0 + {}_nC_1 + \ldots + {}_nC_n = 2^n$.

Using the binomial theorem,

$$2^n = (1 + 1)^n$$
$$= {}_nC_0 \times 1^n + {}_nC_1 \times 1^{n-1} \times 1 + \ldots + {}_nC_n \times 1^n$$
$$= {}_nC_0 + {}_nC_1 + \ldots + {}_nC_n$$

c)

n	Triangular Numbers	Combinatorial Form
1	1	${}_2C_2$
2	3	${}_3C_2$
3	6	${}_4C_2$
4	10	${}_5C_2$
5	15	${}_6C_2$

d) The nth triangular number is ${}_{n+1}C_2$.

Example 6 Factoring Using the Binomial Theorem

Rewrite $1 + 10x^2 + 40x^4 + 80x^6 + 80x^8 + 32x^{10}$ in the form $(a + b)^n$.

Solution

There are six terms, so the exponent must be 5.
The first term of a binomial expansion is a^n, so a must be 1.
The final term is $32x^{10} = (2x^2)^5$, so $b = 2x^2$.
Therefore, $1 + 10x^2 + 40x^4 + 80x^6 + 80x^8 + 32x^{10} = (1 + 2x^2)^5$

Key Concepts

- The coefficients of the terms in the expansion of $(a + b)^n$ correspond to the terms in row n of Pascal's triangle.
- The binomial $(a + b)^n$ can also be expanded using combinatorial symbols:
  $(a+b)^n = {}_nC_0a^n + {}_nC_1a^{n-1}b + {}_nC_2a^{n-2}b^2 + \ldots + {}_nC_nb^n$ or $\sum_{r=0}^{n} {}_nC_ra^{n-r}b^r$
- The degree of each term in the binomial expansion of $(a + b)^n$ is n.
- Patterns in Pascal's triangle can be summarized using combinatorial symbols.

Communicate Your Understanding

1. Describe how Pascal's triangle and the binomial theorem are related.
2. a) Describe how you would use Pascal's triangle to expand $(2x + 5y)^9$.
   b) Describe how you would use the binomial theorem to expand $(2x + 5y)^9$.
3. Relate the sum of the terms in the nth row of Pascal's triangle to the total number of subsets of a set of n elements. Explain the relationship.

Practise

A

1. Rewrite each of the following using Pascal's formula.
   a) ${}_{17}C_{11}$
   b) ${}_{43}C_{36}$
   c) ${}_{n+1}C_{r+1}$
   d) ${}_{32}C_4 + {}_{32}C_5$
   e) ${}_{15}C_{10} + {}_{15}C_9$
   f) ${}_nC_r + {}_nC_{r+1}$
   g) ${}_{18}C_9 - {}_{17}C_9$
   h) ${}_{24}C_8 - {}_{23}C_7$
   i) ${}_nC_r - {}_{n-1}C_{r-1}$

2. Determine the value of k in each of these terms from the binomial expansion of $(a + b)^{10}$.
   a) $210a^6b^k$
   b) $45a^kb^8$
   c) $252a^kb^k$

3. How many terms would be in the expansion of the following binomials?
   a) $(x + y)^{12}$
   b) $(2x - 3y)^5$
   c) $(5x - 2)^{20}$

4. For the following terms from the expansion of $(a + b)^{11}$, state the coefficient in both ${}_nC_r$ and numeric form.
   a) a^2b^9
   b) a^{11}
   c) a^6b^5

Apply, Solve, Communicate

B

5. Using the binomial theorem and patterns in Pascal's triangle, simplify each of the following.

a) ${}_9C_0 + {}_9C_1 + \ldots + {}_9C_9$

b) ${}_{12}C_0 - {}_{12}C_1 + {}_{12}C_2 - \ldots - {}_{12}C_{11} + {}_{12}C_{12}$

c) $\sum_{r=0}^{15} {}_{15}C_r$ **d)** $\sum_{r=0}^{n} {}_nC_r$

6. If $\sum_{r=0}^{n} {}_nC_r = 16\ 384$, determine the value of n.

7. a) Write formulas in combinatorial form for the following. (Refer to section 4.4, if necessary.)

i) the sum of the squares of the terms in the nth row of Pascal's triangle

ii) the result of alternately adding and subtracting the squares of the terms in the nth row of Pascal's triangle

iii) the number of diagonals in an n-sided polygon

b) Use your formulas from part a) to determine

i) the sum of the squares of the terms in row 15 of Pascal's triangle

ii) the result of alternately adding and subtracting the squares of the terms in row 12 of Pascal's triangle

iii) the number of diagonals in a 14-sided polygon

8. How many terms would be in the expansion of $(x^2 + x)^8$?

9. Use the binomial theorem to expand and simplify the following.

a) $(x + y)^7$ **b)** $(2x + 3y)^6$

c) $(2x - 5y)^5$ **d)** $(x^2 + 5)^4$

e) $(3a^2 + 4c)^7$ **f)** $5(2p - 6c^2)^5$

10. Communication

a) Find and simplify the first five terms of the expansion of $(3x + y)^{10}$.

b) Find and simplify the first five terms of the expansion of $(3x - y)^{10}$.

c) Describe any similarities and differences between the terms in parts a) and b).

11. Use the binomial theorem to expand and simplify the following.

a) $\left(x^2 - \frac{1}{x}\right)^5$ **b)** $\left(2y + \frac{3}{y^2}\right)^4$

c) $(\sqrt{x} + 2x^2)^6$ **d)** $\left(k + \frac{k}{m^2}\right)^5$

e) $\left(\sqrt{y} - \frac{2}{\sqrt{y}}\right)^7$ **f)** $2\left(3m^2 - \frac{2}{\sqrt{m}}\right)^4$

12. Application Rewrite the following expansions in the form $(a + b)^n$, where n is a positive integer.

a) $x^6 + 6x^5y + 15x^4y^2 + 20\,x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$

b) $y^{12} + 8y^9 + 24y^6 + 32y^3 + 16$

c) $243a^5 - 405a^4b + 270a^3b^2 - 90a^2b^3 + 15ab^4 - b^5$

13. Communication Use the binomial theorem to simplify each of the following. Explain your results.

a) $\left(\frac{1}{2}\right)^5 + 5\left(\frac{1}{2}\right)^5 + 10\left(\frac{1}{2}\right)^5 + 10\left(\frac{1}{2}\right)^5 + 5\left(\frac{1}{2}\right)^5 + \left(\frac{1}{2}\right)^5$

b) $(0.7)^7 + 7(0.7)^6(0.3) + 21(0.7)^5(0.3)^2 + \ldots + (0.3)^7$

c) $7^9 - 9 \times 7^8 + 36 \times 7^7 - \ldots - 7^0$

14. a) Expand $\left(x + \frac{2}{x}\right)^4$ and compare it with the expansion of $\frac{1}{x^4}(x^2 + 2)^4$.

b) Explain your results.

15. Use your knowledge of algebra and the binomial theorem to expand and simplify each of the following.

a) $(25x^2 + 30xy + 9y^2)^3$

b) $(3x - 2y)^5(3x + 2y)^5$

16. **Application**

a) Calculate an approximation for $(1.2)^9$ by expanding $(1 + 0.2)^9$.

b) How many terms do you have to evaluate to get an approximation accurate to two decimal places?

17. In a trivia contest, Adam has drawn a topic he knows nothing about, so he makes random guesses for the ten true/false questions. Use the binomial theorem to help find

a) the number of ways that Adam can answer the test using exactly four *true*s

b) the number of ways that Adam can answer the test using at least one *true*

ACHIEVEMENT CHECK

Knowledge / Understanding	Thinking / Inquiry / Problem Solving	Communication	Application

18. **a)** Expand $(h + t)^5$.

b) Explain how this expansion can be used to determine the number of ways of getting exactly h heads when five coins are tossed.

c) How would your answer in part b) change if six coins are being tossed? How would it change for n coins? Explain.

C

19. Find the first three terms, ranked by degree of the terms, in each expansion.

a) $(x + 3)(2x + 5)^4$

b) $(2x + 1)^2(4x - 3)^5$

c) $(x^2 - 5)^9(x^3 + 2)^6$

20. **Inquiry / Problem Solving**

a) Use the binomial theorem to expand $(x + y + z)^2$ by first rewriting it as $[x + (y + z)]^2$.

b) Repeat part a) with $(x + y + z)^3$.

c) Using parts a) and b), predict the expansion of $(x + y + z)^4$. Verify your prediction by using the binomial theorem to expand $(x + y + z)^4$.

d) Write a formula for $(x + y + z)^n$.

e) Use your formula to expand and simplify $(x + y + z)^5$.

21. **a)** In the expansion of $(x + y)^5$, replace x and y with B and G, respectively. Expand and simplify.

b) Assume that a couple has an equal chance of having a boy or a girl. How would the expansion in part a) help find the number of ways of having k girls in a family with five children?

c) In how many ways could a family with five children have exactly three girls?

d) In how many ways could they have exactly four boys?

22. A simple code consists of a string of five symbols that represent different letters of the alphabet. Each symbol is either a dot (•) or a dash (–).

a) How many different letters are possible using this code?

b) How many coded letters will contain exactly two dots?

c) How many different coded letters will contain at least one dash?